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ABSTRACT

This is part two of a two-part manual for teachers using SMSG junior high school text materials. For each chapter, a brief overview is given and sample test questions are listed; each section contains a discussion of the topic at hand and answers to all the exercises. Chapter topics include: (1) the rational number system; (2) parallels; (3) triangles; (4) parallelograms; (5) right prisms; (6) circles; (7) statistics and graphs; (8) mathematical systems; and (9) mathematics at work in science. (MP)

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

VOLUME I (Part 2)

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Commentary for Teachers

Volume I (Part 2)

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CHAPTER 8

THE RATIONAL NUMBER SYSTEM II

8-1. Ratios and Proportion

This section should not take more than two class days. Class discussions should include some simple calculations that can be done in a few minutes. Do not spend an extra day or class time for the topic of multiple ratios.

The purpose of the Section 1 is to give a meaningful introduction to ratio and proportion. The important concept of this chapter is proportion. A proportion arises when two quantities are compared by measurement in different situations. When a physical or mathematical law can be written as a proportion, then this law can be used to deduce new information from old. Thus in our first example, shadow length is proportional to height. Knowing the constant of proportionality and the shadow length, the height can be computed. The ideas behind the words ratio, percent, and decimal enable us to express old and familiar properties of rational numbers in a handy way. Indeed, a ratio is a rational number, and so is a percent. The decimal notation is just a notation. Part of our problem is to justify the use of these new words in place of already familiar terms. Naturally, we have to spend a great deal of time in developing sufficient skill with these notations for them to become really handy for our use. Certainly there are important and interesting results to be established. No one will call the theorem that only rational numbers have repeating decimal expansions a trivial result. The correspondence of a point on the number line with a decimal expansion is an exciting and a far-reaching result. On the other hand, in the final analysis, they are simply notations and add only a little to our understanding of rational numbers as a mathematical system. Proportion, however, is a new concept, essentially having its origin in physical examples. If two quantities are proportional, then any two ratios of corresponding values of these quantities are equal since each ratio is equal to the constant of proportionality. Equations, or proportions, of this type are used frequently in our examples.

The examples of this section will provide the students with situations where it is a natural thing to introduce the notations of ratio, percent, and decimals as handy and efficient short cuts of expression. It is our hope that if a purpose is seen for their introduction, the teaching of the technique of their use will prove easier.

Answers to Exercises 8-1-a

1. Depends on height of individuals. Length of shadow is $\frac{2}{3}$ × height.

2.

12	27	$\frac{4}{9}$
36	81	$\frac{4}{9}$
10	$22\frac{1}{2}$	$\frac{4}{9}$
64	144	$\frac{4}{9}$
$13\frac{1}{3}$	30	$\frac{4}{9}$

3. (a) 24 boys.

   (b) Ratio of number of girls to total is $\frac{1}{3}$.

(c) Ratio of number of boys to total is $\frac{2}{3}$.

(d) Ratio of number of girls to number of boys is $\frac{1}{2}$.

4. 72 boys.

5. Such a class is not possible; it would have to have $66\frac{2}{3}$ boys.

6. The following ratios are equal:

(a), (c), (d), (e).

7. (a) $x = 15$
   (b) $x = 42$
   (c) $x = 15$
   (d) $x = 9$
   (e) $x = 56$

*8. (a) EF = 3 in. (b) DE = $7\frac{1}{2}$ in.

9. In the ratio 3:1 we have:

3 cups butter
2 cups sugar
6 eggs
$4\frac{1}{2}$ cups flour
3 teaspoons vanilla

It will now make 180 cookies.

To make 30 cookies you would rewrite the recipe in the ratio 1:2.

$\frac{1}{2}$ cup butter
$\frac{1}{3}$ cup sugar
1 egg
$\frac{3}{4}$ cup flour
$\frac{1}{2}$ teaspoon vanilla

10. 10 inches long.

11. 110 feet tall.

12. $202.50.

Answers to Exercises 8-1-b

	Miles travelled	Time in hours	Ratio of distance to time
1.	20	$\frac{2}{5}$	50
	5	$\frac{1}{10}$	50
	8.3	$\frac{1}{6}$	49.8 (or 50, of $8\frac{1}{3}$ miles are used.)
2.	25 miles	$\frac{1}{2}$ hour	50 m.p.h.
	$22\frac{1}{2}$ miles	$\frac{3}{4}$ hour	30 m.p.h.
	1 mile	$\frac{1}{80}$ hour	80 m.p.h.

3. (a) $12\frac{1}{2}$ m.p.h. (c) 0 m.p.h.
   (b) $37\frac{1}{2}$ m.p.h. (d) $66\frac{2}{3}$ m.p.h.

4. (a) $1.65
   (b) 45 cents (45 cents per dozen)
   (c) $119.
   (d) 6 × 52.8 = 316.8 feet, or 317 to the nearest foot.

*5. $= \frac{528}{12}$ or 44 feet per second.

*6. Mach 1 is the speed of sound, Mach 2 is twice the speed of sound, Mach 3 is three times the speed of sound, etc. Thus, a speed of 0.7 Mach denotes a ratio: the ratio of your speed to the speed of sound is $\frac{7}{10}$.

The speed of sound changes with the altitude (it is slower at high altitudes) and so a Mach number is not an absolute unit (like 300 m.p.h.) but instead it changes depending on how high you are.

At sea level the speed of sound is 1100 feet per second.

7. (a)	12	14	$\frac{12}{14}$	$\frac{6}{7}$
(b)	18	21	$\frac{18}{21}$	$\frac{6}{7}$
(c)	30	35	$\frac{30}{35}$	$\frac{6}{7}$
(d)	$85\frac{5}{7}$	100	$\frac{8[illegible]}{10[illegible]}$	$\frac{6}{7}$
(e)	100	$116\frac{2}{3}$	$\frac{100}{116\frac{2}{3}}$	$\frac{6}{7}$

8. Area = $\frac{1}{2}$ base × height, so constant of proportionality is $\frac{1}{2}$.

*Exercises 8-1-c

This is a starred section, to be used with some students if time permits. To make 24 pounds of a mixture of peanuts, cashews, and pecans in a ratio of 5:2:1 the grocer would use 15 pounds of peanuts, 6 pounds of cashews, and 3 pounds of pecans. The answers to the questions in the discussion are

1. 8 pounds of nuts.
2. 5:8 is the ratio of peanuts to total.
3. 15 pounds of peanuts.
4. Cashews 2:8 is ratio to total
   4 pounds of cashews.
5. Pecans 1:8 is ratio of pecans to total
   2 pounds of pecans.

Answers to Exercises 8-1-c

1. 35 pounds of peanuts, 14 lbs. of cashews, 7 lbs. of pecans.
2. 50 pounds peanuts, 30 pounds cashews, 20 pounds pecans.
3. 20 pounds cashews, 10 pounds pecans, 60 pounds in all.
4. 24 inches.

***

8-2. Percent

The concepts of percent are introduced briefly in Section 2. In Section 5 of this chapter, percent will be treated more fully. In both of the sections the meaning of percent is based on the idea that "a" % means $\frac{a}{100} = a \times \frac{1}{100}$. All three "cases" of percent are introduced informally in Section 2 with numbers that are easily handled. However, you will notice that the three "cases" of percent are not referred to in this textbook. Instead, the emphasis is on the concepts of 100% and 1%. Notice that the solutions of all problems are set up in the form $\frac{a}{b} = \frac{x}{100}$. The method of solving for x should be the method that the pupil understands. Pupils should be encouraged to use the more sophisticated solution if they understand it.

In the example of the class of 11 girls and 14 boys, the fraction $\frac{44}{100}$ indicates percent easily because the denominator is 100. The sum of $\frac{11}{25}$ and $\frac{14}{25}$ is one. The sum of the ratios $\frac{44}{100}$ and $\frac{56}{100}$ is one, and the sum of 44% and 56% is 100% or one.

Exercises 8-2-a emphasize writing percents first as fractions with 100 as the denominator, and then with the % symbol. Another point of emphasis is that 100% stands for one or the whole of a quantity.

Answers to Exercises 8-2-a

1. (a) $\frac{1}{10}$ (c) 20 (e) 35 (g) $\frac{3}{10}$ (i) $\frac{1}{20}$
   (b) 10% (d) $\frac{1}{5}$ (f) $\frac{7}{20}$ (h) 30% (j) 5%
2. One
3. 100%

4. (a) 50% (c) 75% (e) 40% (g) 80% (i) 100%
(b) 25% (d) 20% (f) 60% (h) 100% (j) 140%

5. (a) $\frac{12}{50}$ or $\frac{6}{25}$ or $\frac{24}{100}$, 24% ; $\frac{10}{50}$ or $\frac{1}{5}$ or $\frac{20}{100}$, 20% ;
$\frac{15}{50}$ or $\frac{3}{10}$ or $\frac{30}{100}$, 30% ; $\frac{13}{50}$ or $\frac{26}{100}$, 26% .
(b) 1 (c) 100%
(d) The ratios total 1, and the percents total 100%.

6. (a) Mortgage 20% ; taxes 5% ; car 9% ; food 30% ; clothing 12% ; operating 8% ; health and recreation 6% ; savings and insurance 10%.
(b) The 8 answers total 100% .

The second lesson in 8-2 emphasizes the use of percent for purposes of comparison, and for giving information in more useable form. The use of 1% and 10% in solving problems is introduced here. Pupils should learn to calculate mentally 1% of a quantity, and also 10% . Frequent oral practice should include such calculations.

Answers to Exercises 8-2-b

1. (a) 60% (b) 40% (c) 100%

2. (a) 50% or $\frac{50}{100}$ (b) decreased

3. (a) 65% or $\frac{65}{100}$ (b) higher than both

4. (a) 30 (b) 5% (c) 6 (d) 60

5. 6%

6. (a) 25% (b) 75%

7. (a) $\frac{1}{10}$ (b) 10%

8. 125%

9. $\frac{1}{2}$ or $\frac{1}{2}$ of 1%.

10. 550 hint $\frac{22}{x} = \frac{4}{100}$

8-3. Decimal Notation

Section 3 should take at most 3 days. The main purposes are (a) to review base and place, (b) to explain the notation for the decimal system, (c) to present an explanation for changing a rational number to a decimal, (d) to review the number line, (e) to provide practice in changing rational numbers to decimals, and (d) to introduce rounding numbers.

The explanation of changing a rational number to a decimal is motivated by the number line and the measurement of a segment of the number line. The main point here is that when we wish to increase the precision of our measurement, a new unit one-tenth of the previous unit is selected. Our positional system of writing numbers is analagous to this since reading from left to right the value of each place decreases by a factor of 10. If we used a unit $\frac{1}{2}$ of the previous unit and continued using powers of $\frac{1}{2}$, the number would be written in base two; if we used a unit $\frac{1}{12}$ of the previous unit and continued using powers of $\frac{1}{12}$, the number would be written in base twelve.

When the class finds the measure of AB it might be highly desirable to introduce the meter stick and show the powers of 10 and one-tenth used in the metric system.

The teacher could illustrate changing a rational number into other bases.

Example 1. Write $\frac{1}{4}$ as a duodecimal.

There are 12 twelfths in one unit, so in $\frac{1}{4}$ of a unit, there are $\frac{12}{4}$, or 3 twelfths, and none left over. $\frac{1}{4} = 0.3_{\text{twelve}}$.

Example 2. Write $\frac{5}{8}$ as a duodecimal.

There are 12 twelfths in one unit and 60 twelfths in 5 units, so in $\frac{5}{8}$ of a unit, there are $\frac{60}{8}$, or $7\frac{1}{2}$ twelfths. The twelfth's digit is "7" and there is $\frac{1}{2}$ of a twelfth left over. There are 12 one-hundred-forty-fourths in one twelfth of a unit, so in $\frac{1}{2}$ of a twelfth, there are $\frac{12}{2}$, or 6 one-hundred-forty-fourths and none left over. The one-hundred-forty-fourth's digit is "6".

$\frac{5}{8} = 0.76_{\text{twelve}}$

Example 3. Write $\frac{3}{7}$ in base two.

There are 2 halves in one unit and 6 halves in 3 units, so in $\frac{3}{7}$ of a unit, there are $\frac{6}{7}$ left over.

There are 2 fourths of a unit in one half of a unit and 12 fourths of a unit is 6 halves of a unit, so in $\frac{6}{7}$ of a half, there are $\frac{12}{7}$, or $1\frac{5}{7}$ fourths of a unit. The fourth's digit is "1" and there is $\frac{5}{7}$ of a fourth left over.

There are 2 eighths of a unit in one fourth of a unit and 10 eighths of a unit in 5 fourths of a unit, so in $\frac{5}{7}$ of a fourth, there are $\frac{10}{7}$, or $1\frac{3}{7}$ eighths of a unit. The eighth's digit is "1" and there is $\frac{3}{7}$ of an eighth left over.

But now we start to repeat, since $\frac{3}{7}$ is the fraction with which we started. Thus, $\frac{3}{7} = 0.011011011\ldots_{\text{two}}$

Answers to Exercises 8-1-A

1. (a) 66.07
   (b) 130.10
   (c) 50.24
   (d) 10.483
   (e) 0.0800

2. (a) $5(10) + 2(1) + 5\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^2}\right)$

(b) $1(1) + 2\left(\frac{1}{10}\right) + 1\left(\frac{1}{10^2}\right) + 3\left(\frac{1}{10^3}\right)$

(c) $1\left(\frac{1}{10}\right)$

(d) $3(1) + 1\left(\frac{1}{10^2}\right)$

(e) $1\left(\frac{1}{10^2}\right) + 1\left(\frac{1}{10^4}\right)$

3. (a) Seven and two hundred thirty-six thousandths;
(b) Four thousandths;
(c) Three hundred sixty and one-hundred one thousandths;
(d) One and one-hundred one ten thousandths.

4. (a) 300.52 (c) 0.014
(b) 0.0507 (d) 60.07

5*. 0.5

6*. 0.6_{twelve}

7*. $0.T_{\text{twelve}}$

8*. $10.011_{\text{two}} = 1(2) + 1\,\frac{1}{2^2} + 1\,\frac{1}{2^3} =$

$= 2 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{2}{8} + \frac{1}{8} =$

$= 2\frac{3}{8} = 2.375.$

Answers to Exercises 8-3-b

1. (a) 1.75 (c) 0.125 (e) 0.88
(b) 1.8 (d) 1.875

2. (a) 0.11 (c) 0.88 (e) 0.33
(b) 0.67 (d) 5.11

3. (a) 26.4 (c) 108.0 (e) 0.2
   (b) 48.7 (d) 0.0 (f) 1.2

4. (a) 4.049 (b) 37.307 (c) 0.001 (d) 0.000

5. (a) 0.4 ; 0.43 ; 0.429
   (b)

This is magnified 5 times from the sketch in the pupil's text.
(i) 0.429 is closest to $\frac{3}{7}$.
(ii) 0.4 is least like $\frac{3}{7}$.

6. (a) $\frac{3}{7}$ = 0.43

$\frac{17}{42}$ = 0.40

Rank $\frac{3}{7}$ first

(b) $\frac{5}{16}$ = 0.3125

$\frac{26}{83}$ = 0.313

Rank $\frac{26}{83}$ first

7*. (a) $1(1) + \left(1\ \frac{1}{2}\right) = 1.1_{\text{two}}$

(b) $1\left(\frac{1}{2^2}\right) = 0.01_{\text{two}}$

(c) $0(1) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2^2}\right) + 0\left(\frac{1}{2^3}\right) + 1\left(\frac{1}{2^4}\right)$ and so on.

$\frac{1}{3} = 0.010101\ldots_{\text{two}}$

(d) $\frac{1}{3} = 4\left(\frac{1}{12}\right)$

$\frac{1}{3} = 0.4_{\text{twelve}}$

(e) Both. The fact that a number can be expressed as terminating, repeating, or an infinite expansion is a property of the base and the number itself.

(f) Six, nine, or any other number which contains 3 as a factor.

8. The sum of the series is $\frac{1}{3}$.

See 7(c) above.

$$0.010101_{\text{two}}\ldots = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} \text{ and so on.}$$

8.4. Arithmetic Operations with Decimals

This section develops the arithmetic operations (addition, subtraction, multiplication and division) for terminating decimals. We are not in a position to do this for non-terminating decimals. For example,

$$0.4444\ldots + 0.7777\ldots$$

can be added as decimals (without converting them to fractions) only by rounding them off to a fixed number of places, in other words, by replacing each of them by a terminating decimal. A full discussion of this would involve studying the error made when we round numbers. This is beyond the scope of the present chapter.

Which decimals terminate? First of all, any terminating decimal is a repeating decimal (since 0 is repeated), and, therefore, represents a rational number. But not all rational numbers have terminating decimals; for example, $\frac{2}{7} = .285714285714285\ldots$ is not terminating. The terminating decimals are precisely those rational numbers that can be written with denominator a power of ten. For example, $\frac{1}{5}$ does not have a denominator that is a power of ten, but it can be written in that form:

$$\frac{1}{5} = \frac{2}{10} = 0.2000000000\ldots$$

So the concept of terminating decimal is a special one, that depends on the particular base being used. Thus, if we used the Base 7, all fractions that could be written with denominator a power of seven would give rise to terminating decimals.

Thus, the concept of the infinite decimal expansion is the really important concept, while the terminating decimal is a special concept that depends on the particular base being used. However, even with the infinite decimal there is some ambiguity. Thus, $\frac{1}{2}$ has two different representations as a decimal:

$$\frac{1}{2} = 0.500000000 \ldots = 0.4999999999 \ldots$$

The decimals for which this ambiguity arises are precisely the terminating decimals. If a decimal terminates (i.e., has all zeros from some point on), then it can be written with all nine from some point on.

The purpose of this section is to introduce the arithmetic operations on decimals, and for this purpose it was thought best to work with terminating decimals, and to postpone the study of non-terminating decimals (real numbers) to a later chapter.

In the section on rounding it could be pointed out by the teacher that we can estimate the answer to a complicated problem by means of rounding. Thus, for example, an estimate of

$$.7483 \quad + \quad .3451$$

could be obtained by rounding each number to one place, giving

$$.7 + .3 = 1.0 .$$

Or again in multiplication,

$$.25 \times 8.34$$

is approximately equal to

$$\frac{1}{4} \times 8 = 2 .$$

***

Answers to Exercises 8-4-a

1. (a) 1.50 (b) 1.101 (c) 1.00302 (d) 23.30

2. (a) .08 (b) .075 (c) .375 (d) 1.0045

3. (a) .415 (b) .1635

4. No. They want a total of 50.4 feet of wire.

5. 26.05 pounds of sugar are left.

6. $\frac{7}{16} = 0.4375$, and, therefore, 0.45 lb. is greater than 7 ounces.

7. $$\begin{array}{r} 10.01_{two} \\ +\ 1.01_{two} \\ \hline 11.10_{two} \end{array} \qquad 11.1_{two} = 1(2^1) + 1 + \frac{1}{2} = 3\frac{1}{2} \text{ or } 3.5$$

Answers to Exercises 8-4-b

1. (a) 0.00081 (c) 144.
   (b) 0.00625 (d) 0.027270

2. (a) 0.0003 (c) 1.4375
   (b) 0.3 (d) 255.

3. 375, 37.5, 3.75, 0.375, 0.0375, 0.00375.

4. 0.0625.

5. 397.21.

6. 3.102_{seven} (In the multiplication table to the base seven we have: $4_{seven} \times 4_{seven} = 22_{seven}$, and

   $2_{seven} \times 4_{seven}$ [illegible] 11_{seven} .

Answers to Exercises 8-4-c

1. (a) 0.04
   (b) 0.04
   (c) 0.00 (or: 0.01. The rule about 5 would give: .005 rounds to .00. However, 51 is more than half of 100, and so .0051 really should round to .01.)
   (d) 0.02
2. (a) 0.160 (c) 0.000
   (b) 0.001 (d) 0.325
3. (a) 0.375 (b) 0.250 (c) 0.667
4. (a) 0.3 (c) 0.1
   (b) 0.3 (d) 0.0
5. (a) 43.29 sq. rd. (b) 43.8916 sq. rd.
   0.6016 sq. rd. difference

8-5. Percent

In this section the concept of percent is developed in terms of decimals as well as in the form of $\frac{c}{100} = c \times \frac{1}{100} = c\%$. In studying the fractional and decimal equivalents for percent, it is emphasized that the equivalents are different names for the same number.

$$72\% = \frac{72}{100} = .72 = \frac{18}{25}$$

$$.15 = \frac{15}{100} = 15\% = \frac{3}{20}$$

$$\frac{1}{5} = \frac{20}{100} = .20 = 20\%$$

$$\frac{3}{4} = \frac{75}{100} = .75 = 75\%$$

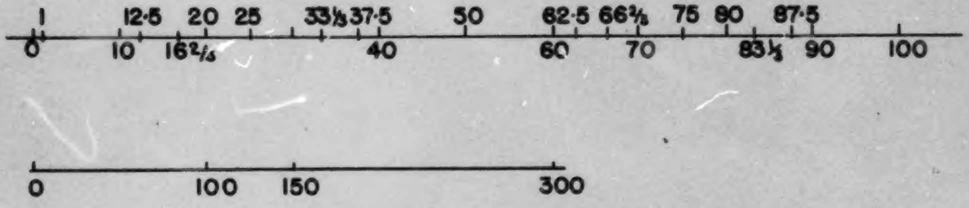
Notice that percents such as $33\frac{1}{3}\%$ when given in decimal form do not include the $\frac{1}{3}$. In computation, problems involving percents such as $33\frac{1}{3}\%$ should be solved by using the fraction rather than the decimal.

Answers to Exercises 8-5-a

1.	(b)	$\frac{25}{100}$	.25	25%
	(c)	$\frac{3}{4}$	.75	75%
	(d)	$\frac{1}{5}$	$\frac{20}{100}$	20%
	(e)	$\frac{2}{5}$	$\frac{40}{100}$	.40
	(f)	$\frac{3}{5}$	.60	60%
	(g)	$\frac{80}{100}$	.80	80%
	(h)	$\frac{1}{3}$	$\frac{33\frac{1}{3}}{100}$	$33\frac{1}{3}\%$
	(i)	$\frac{7}{10}$	.70	70%
	(j)	$\frac{2}{3}$	$\frac{66\frac{2}{3}}{100}$	$66\frac{2}{3}\%$
	(k)	$\frac{30}{100}$	.30	30%
	(l)	$\frac{1}{10}$	$\frac{10}{100}$	10%
	(m)	$\frac{9}{10}$	$\frac{90}{100}$	.90
	(n)	$\frac{12\frac{1}{2}}{100}$	.125	$12\frac{1}{2}\%$ or 12.5%
	(o)	3	3	300%
	(p)	$\frac{3}{8}$	$\frac{37\frac{1}{2}}{100}$	$37\frac{1}{2}\%$ or 37.5%

(q)	$\frac{3}{2}$	$\frac{150}{100}$	1.5
(r)	$\frac{5}{8}$	.625	62.5% or $62\frac{1}{2}\%$
(s)	$\frac{1}{100}$	$\frac{1}{100}$	1%
(t)	$\frac{87.5}{100}$	.875	87.5% or $87\frac{1}{2}\%$
(u)	1	$\frac{100}{100}$	1
(v)	$\frac{1}{6}$	.166 ...	$16\frac{2}{3}\%$
(w)	$\frac{83\frac{1}{3}}{100}$	.833 ...	$83\frac{1}{3}\%$

2.



4. (a) $\frac{8}{25}$ (b) $\frac{9}{10}$ (c) $\frac{6}{5}$ or $1\frac{1}{5}$

5. (a) 52% (b) 35% (c) 95% (d) 30%

The study of percent is valuable if the pupil can use the concept to solve problems from everyday experience, and can understand and interpret data expressed in percent. Pupils are given all three cases of percent from every day situations. The examples of solutions all follow the pattern $\frac{a}{b} = \frac{c}{100}$. In class discussion the estimation of a reasonable answer helps the pupil to understand the relationships among a, b, and c in various situations, as well as serving as a check on an answer.

Property 9 from Chapter 6 is used frequently in the solution of problems in this section. Property 9 states, "If the numerator and denominator of a fraction are rational numbers, the number represented by the fraction is not changed by multiplication of numerator and denominator by the same non-zero rational."

Answers to Exercises 8-5-b

1. (a) 40 (b) 32 (c) 20 (d) 26
2. (a) 75% (b) 90% (c) 45% (d) 95%
3. (a) $.70 (b) $1.98
4. $875.
5. $180,000.
6. 30%
7. Yes; the two commissions, $6580. and $4820. total $11,400.
8. (a) $4.38 ($5.98—$1.60) (b) $2.48 ($3.40—$.92)
9. (a) 1117 (b) 34% (c) 34% (d) 32% (e) 100%
10. $2400.

Fractional percents are developed so that percent can be expressed to the nearest tenth of one percent. In this lesson the concepts of percent of increase and percent of decrease also are presented. If a pupil has difficulty with percent of increase or decrease, he should be encouraged to make drawings to illustrate. The number used as a standard for comparison is always the earlier by the calendar or clock.

Answers to Exercises 8-5-c

1. (a) 11.4% (c) 36.9% (e) 8.0%
   (b) 21.0% (d) 22.7% (f) 100.0%
2. 20%
3. 10%
4. 1550 (1240 + 310)
5. 32.1%
6. (a) $\frac{1}{2}$% (b) $4,237.50
7. 8.3%
8. Pupils may be interested in comparing percents of increase.
9. (a) Jones .301; Smith .293; (b) Jones
10. 28.1%

The second lesson on percent of increase or decrease emphasizes the two approaches to such a problem. In a problem involving percent of increase, one method finds the actual increase by subtraction, then the increase is expressed as a percent of the earlier quantity. In the second method the percent that the later quantity is of the earlier quantity is computed first. Then 100% is subtracted to give the percent of increase. Pupils need to realize, for example, that an increase of 15% in a quantity always gives a quantity that is 115% of the original; and, also, a decrease of 15% in a quantity always gives a quantity that is 85% of the original. A few minutes of oral work with the class with these ideas would be helpful.

Answers to Exercises 8-5-d

1. (a) 150 (b) 120 (c) decrease of 20%
2. 14.3%
3. 5.6%
4. (a) 15 lb. 2 oz. (b) 124.8%
5. 30.9%
6. (a) The 1960 wages are less than the 1958 wages.

   (b) 96%

   Suggestion: Start with 1958 wages of $100. per week. Some pupils will be interested in finding what % of increase from 1959 to 1960 will bring the wages back to $100. 25%
7. (a) 15% (b) 17.6%

8-6. Decimal Expansion

May every rational number be named by a decimal numeral? Does every decimal numeral name a rational number? These two key questions provide the motivation for this section.

We have made no attempt to present any rigorous arguments to answer the first question in the affirmative. We did attempt to provide sufficiently many suggestive exercises through which the pupils can proceed by induction to provide their own affirmative answer.

The negative answer to the second question has the effect of keeping the door open for a discussion of irrational numbers and, of course, eventually the real numbers.

We have attempted to provide for a transition to a discussion of the real number system by introducing the notion of a decimal expansion.

The pupils should have ample practice in keeping their computational skills sharp. If you wish to provide more than we have allowed for, please feel free to make any necessary adjustments to meet your classes individual needs.

The procedure for finding a fraction to name a rational number which is expressed as a decimal expansion is particularly important. A similar procedure on a more sophisticated level is used to prove that every repeating decimal expansion names a rational number.

BRAINBUSTER. If $\frac{a}{b}$ can be expressed as a fraction whose denominator is a power of 10, then it can be expressed as a terminating decimal numeral (from our previous discussion).

If it cannot, then the remainders obtained by dividing a by b must be of the following type $0 < r < b$. If r assumes all counting number values within this interval, prior to the (b-1)st remainder, then it will of necessity repeat one of the values at the (b-1)st remainder. It may, of course, repeat one of these values within the interval prior to the (b-1)st remainder.

Therefore, the rational number $\frac{a}{b}$ may be named by a decimal numeral which either ends or repeats after (b-1) remainders at most.

Answers to Questions in Class Discussion

1. 1, 4, 2, 8, 5, 7
2. No.
3. Yes
4. $0.0909\overline{09}$ ....
5. After the first subtraction.
6. No; by dots.
7. Yes; $0.0909\overline{09}$ ....

Answers to Exercises 8-6-a

1. $0.0769230\overline{769230}$ ....
   - (a) After the 5th subtraction.
   - (b) No.
   - (c) 0.769230769230.
   - (d) Yes
   - (e) $0.769230\overline{769230}$ ....
2. (a) $0.33\overline{3}$ ...
   - (b) $0.16\overline{6}$ ...
   - (c) $0.11\overline{1}$ ...

Answers to Exercises 8-6-b

1. (a) $0.0909\overline{09}$ ...
   - (b) $0.1818\overline{18}$ ...
   - (c) $0.2727\overline{27}$ ...
   - (d) $0.8181\overline{81}$ ...
   - (e) $1.2727\overline{27}$ ...
   - (f) $2.0909\overline{09}$ ...
2. (a) the decimal numeral naming $\frac{2}{11}$ is twice the decimal numeral naming $\frac{1}{11}$.
   - (b) the decimal numeral naming $\frac{3}{11}$ is three times the decimal numeral naming $\frac{1}{11}$.

     the decimal numeral naming $\frac{9}{11}$ is nine times the decimal numeral naming $\frac{1}{11}$.

     the decimal numeral naming $\frac{14}{11}$ is fourteen times the decimal numeral naming $\frac{1}{11}$.
3. $0.4545\overline{45}$ ...
4. $0.6363\overline{63}$ ...
5. Yes.

Answers to Exercises 8-6-c

1. (a) 0.5 (e) 0.6 (i) $0.6\overline{66}...$
   (b) 0.2 (f) 0.125 (j) $0.0909\overline{09}...$
   (c) 0.75 (g) 0.8 (k) $0.5\overline{55}...$
   (d) 0.6 (h) 0.375 (l) $0.\overline{1764705882352941}...$
2. a, b, c, d, e, f, g, h,
3. i, j, k, l.
4. (a) 0.2, 0.4, 0.8 (e) 0.05, 0.15, 0.55
   (b) 0.1, 0.4, 0.7 (f) 0.02, 0.84, 0.94
   (c) 0.125, 0.375, 0.875 (g) 0.001, 0.112, 0.927
   (d) $0.0909\overline{09}...$, $0.4545\overline{45}...$, $0.8181\overline{81}...$

Answers to Discussion Questions

1. (a) 6 (b) 4 (c) 5 (d) 1
2. In each remainder that repeats there will have to be a digit in the quotient that repeats since this digit is a factor of a partial product previously computed.
3. If it were 7 or more, then our quotient would have to be increased by at least 1. Our quotient would be too small.
4. 17
5. Terminating.
6. There are only 17 possible remainders. Therefore, at least by the 18th place one remainder will have to have appeared twice.
7. 17
8. Yes.

Answers to Exercises 8-6-d

1. (a) $0.66\overline{6}\ldots$
   (b) 0.625
   (c) $0.2727\overline{27}\ldots$
   (d) $0.416\overline{6}\ldots$
   (e) $0.288\overline{8}\ldots$
   (f) $0.2\overline{285714}$

2. (a) $0.11\overline{1}\ldots$
   (b) $0.22\overline{2}\ldots$
   (c) $0.33\overline{3}\ldots$
   (d) $0.44\overline{4}\ldots$
   (f) $0.66\overline{6}\ldots$
   (g) $0.77\overline{7}\ldots$
   (h) $0.88\overline{8}\ldots$
   (i) 1.0 (Note that 0.999... approaches 1 and is also a good answer.)

3. (a) $0.\overline{142857}\ldots$
   (b) $0.\overline{285714}\ldots$
   (c) $0.\overline{428571}\ldots$
   (d) $0.\overline{571428}\ldots$
   (e) $0.\overline{714285}\ldots$
   (f) $0.\overline{857142}\ldots$
   (g) $0.999999\ldots$ or 1. See answer to 1(i).

   Yes, for example, the decimal expansion for $\frac{2}{9}$ is twice that for $\frac{1}{9}$, etc. The decimal expansion for $\frac{2}{7}$ is twice that for $\frac{1}{7}$, etc.

BRAINBUSTER. Start off discussion with the following:

Given the fraction $\frac{a}{b}$, where a is a whole number and $b = 10^n$, then $\frac{a}{b}$ can be expressed as a terminating decimal numeral, namely, $a = 10^{-n}$. If a fraction, say $\frac{x}{2^a}$ has as its denominator a power of 2 only,

(a) then there exists a power of 10 which is divisible by 2^a, namely,

$$2^a \cdot 5^a = (2 \cdot 5)^a = 10^a.$$

$\because \frac{x}{2^a}$ can be expressed as a terminating decimal numeral.

(b) If a fraction, say $\frac{y}{5^b}$, has as its denominator a power of 5 only, then there exists a power of 10 which is divisible by 5^b, namely,

$$2^b \cdot 5^b = (2 \cdot 5)^b = 10^b.$$

$\therefore \frac{y}{5^b}$ can be expressed as a terminating decimal numeral.

(c) If a fraction, say $\frac{z}{2^a \cdot 5^b}$, has as its denominator a product of powers of 2 and 5 only, then there exists a power of 10 which is devisible by $2^a \cdot 5^b$, namely,

$$(2^a \cdot 5^b)(2^b \cdot 5^a) = (2^{a+b})(5^{a+b}) = (2 \cdot 5)^{a+b} = 10^{a+b}.$$

$\frac{z}{2^a \cdot 5^b}$ can be expressed as a terminating decimal numeral.

(d) If a fraction, say $\frac{p}{g}$, does not have as its denominator a number which is factorable into powers of 2 only, or powers of 5 only, or a product of powers of 2 and 5 only, then it cannot be expressed as a terminating decimal numeral. For if it could be so expressed, then there would exist some power of 10 which is divisible by this denominator which does not have powers of 2 only or powers of 5 only or a product of powers of 2 and 5 only. But such a power of 10 does not exist since a power of 10 is factorable into only powers of 2 and 5. $\therefore$ the fraction $\frac{p}{g}$ cannot be expressed as a terminating decimal numeral.

The students might say that the digit, zero, is annexed to each previous group of zero's and then a one placed after the new group of zero's. This is probably as good an answer as any at this stage. It does not end if we continue the above procedure. It does not repeat if we continue the above procedure.

Answers to Exercises 8-6-e

1. (a) $\frac{13}{99}$

(b) $\frac{5}{11}$

(c) $\frac{5}{6}$

(d) $\frac{28}{33}$

(e) $\frac{22}{3}$

(f) $\frac{29}{3}$ or $9\frac{2}{3}$

(g) $\frac{136}{33}$ or $4\frac{4}{33}$

(h) $\frac{222}{99}$ or $2\frac{8}{33}$

(i) $\frac{15161}{9999}$

(j) $x = \frac{4714281}{999,999} = \frac{174603}{37037}$

2. (a)

```
      0.13...
  99)13.0000
      9 9xx
      3 10
      2 97
        130
```

(b)

```
      0.45...
  11)5.0000
     4 4xx
       60
       55
        50
```

(c)

```
     0.83...
   6)5.000
     4 8xx
       20
       18
        20
```

(d)

```
      0.84...
  33)28.000
     26 4xx
      1 60
      1 32
        280
```

(e)

```
      7.3...
   3)22.00
     21 xx
      1 0
        9
        10
```

(f)

```
      9.6...
   3)29.00
     27 xx
      2 0
      1 8
        20
```

(g)

```
       4.12...
  33)136.000
     132 xxx
       4 0
       3 3
         70
         66
          40
```

(h)

```
       2.24...
  99)222.000
     198 xxx
      24 0
      19 8
       4 20
       3 96
         240
```

(i)

```
           1.5162...
  9999)15161.00000
        9999 xxxxx
        5162 0
        4999 5
         162 50
          99 99
          62 510
          59 994
           2 5160
           1 9998
             51620
```

```
(j)           4.714285...
       999,999)4714281.000000
               3999996 xxxxxx
                714285 0
                699999 3
                 14285 70
                  9999 99
                  4285 710
                  3999 996
                   285 7140
                   199 9998
                    85 71420
                    79 99992
                     5 714280
                     4 999995
                       714285
```

8-7. Metric System

"The invention of the Hindu-Arabic decimal number system is one of man's outstanding achievements. With it, for the first time in history, masses were able to learn the act of computation. Later Simon Stevin still further simplified the processes of computation by the introduction of the decimal fraction. Today, the decimal fraction should be called the common fraction, so widely is it used in commerce and technology.

"Still later came the metric system of measures, based upon the units, meter, liter, and gram, which are also decimal. ....If the selection of a system of measures were optional with educators, they would unhesitatingly choose a decimal system. They are well aware of the tremendous efforts required to learn, for example, the relationship between linear units in our system: 1 inch $= \frac{1}{12}$ foot, 1 foot $= \frac{1}{3}$ yard, 1 yard $= \frac{2}{11}$ rod, 1 rod $= \frac{1}{320}$ mile. In contrast, they appreciate the simplicity and ease with which the pupil could learn: 1 millimeter = 0.1 centimeter, 1 centimeter = 0.01 meter and 1 meter = 0.001 kilometer.

"From the point of view of teaching and learning, it would not be easy to design a more difficult system than the English system. In contrast, it would seem almost impossible to design a system more easily learned than the metric system."[1]

The above quotation from the Twentieth Yearbook of the National Council of Teachers of Mathematics served as a motivation for this section.

The brief historical sketch is intended to help the pupils see the decimal foundation and origin of the metric system.

Since this is probably the pupils' first formal introduction to the metric system, we urge you to have them "live metric" for a few days. Postpone any attempts at converting from metric to English measures or vice versa. The pupils should have a metric ruler to do some of the problems in this section. We suggest that you use the metric system frequently in subsequent chapters in which measurements are required.

Because of the length of this chapter we were not able to introduce the centigrade scale for temperature measurements. You will also note that we have not developed in any detail the two systems now in vogue, namely, the MKS (meter, kilogram, second) or the CGS (centimeter, gram, second). As a matter of fact, we borrowed from both systems in order to make our tables more easily understandable.

We highly recommend the Twentieth Yearbook of the National Council of Teachers of Mathematics as a rich reference text for both your professional library and your pupils' mathematics library.

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[1] Clark, John R., "A Note on the Yearbook", The Metric System of Weights and Measures, Twentieth Yearbook of National Council of Teachers of Mathematics, Bureau of Publications, Teachers College, Columbia University, New York, 1948.

Answers to Exercises 8-7-a

1. (a) 3 km + 0 hm + 3 dkm + 0 m + 3 dm + 0 cm + 3 mm.
   (b) 2 km + 4 dkm + 5 m + 3 dm + 6 cm.
   (c) 5 m + 3 dm + 4 cm + 2 mm.
   (d) (0 m) + 5 dm + 6 cm + 4 mm.
   (e) (0 m + 0 dm) + 4 cm + 3 mm.

2. (a) 10 (c) 1,000 (e) 100,000
   (b) 100 (d) 10,000 (f) 1,000,000

3. (a) 1500 (c) 25.4 (e) $\frac{1}{2}$ or .5
   (b) 325 (d) 20,000

4. Approximate values are:
   (a) 2(6 cm + 3 cm) ≈ 18 cm
   2(58 mm + 32 mm) ≈ 180 mm
   (b) 3 cm + 3 cm + 5 cm + 3 cm + 4 cm ≈ 18 cm
   33 mm + 26 mm + 50 mm + 26 mm + 38 mm ≈ 173 mm
   (c) 2(7 cm + 5 cm) ≈ 24 cm
   2(65 mm + 50 mm) ≈ 230 mm
   (d) 5 cm + 7 cm + 8 cm ≈ 20 cm
   50 mm + 65 mm + 83 mm ≈ 198 m
   (e) 2(4 cm) + 4(3 cm) ≈ 20 cm
   2(39 mm) + 4(25 mm) ≈ 178 mm
   (f) 8 cm + 5 cm + 4 cm + 3 cm + ≈ 20 cm
   80 mm + 50 mm + 42 mm + 26 mm ≈ 198 mm

5. (a) 2(58 mm + 32 mm) ≈ 180 mm
   (b) 33 mm + 26 mm + 50 cm + 26 mm + 38 mm ≈ 173 mm
   (c) 2(65 mm + 50 mm) ≈ 230 mm
   (d) 50 mm + 65 mm + 83 mm ≈ 198 mm
   (e) 2(39 mm) + 4(25 mm) ≈ 178 mm
   (f) 80 mm + 50 mm + 42 mm + 26 mm ≈ 198 mm

6. The millimeter

7. (b) a centimeter (d) a millimeter (e) a meter

8. (a) 2.54 cm
   (b) 10 mm
   (c) 5 cm
   (d) 25.4 mm
   (e) 50 mm
   (f) 10 cm
   (g) 100 mm

*9. (a) 40,000,000 m
   (b) 4,000,000 dkm
   (c) 400,000 hm
   (d) 40,000 Km
   (e) 400,000,000 dm
   (f) 4,000,000,000 cm
   (g) 40,000,000,000 mm

Research Problem

(a) $\frac{1}{10,000}$ of a cm. (b) $\frac{1}{100,000,000}$ of a cm.

(c) The measurement of a meter is 1,553,164.13 wave lengths of the red light emitted by a cadmium vapor lamp excited under certain specified conditions. This gets the meter down to an accuracy of 1 part in 10,000,000.

Answers to Exercises 8-7-b

1. (a) 10,000 (c) $\frac{1}{100}$ (e) $\frac{1}{1,000,000}$
   (b) 1,000,000 (d) $\frac{1}{10,000}$

2. Measurement of length $\approx$ 6 cm.; Measurement of width $\approx$ 5 cm.
   The Area of the interior $\approx 30 \text{ cm}^2$.

3. Measurement of length $\approx$ 63 mm.; Measurement of width $\approx$ 47 mm.
   The Area of the interior $\approx (63)(47) \text{ mm}^2 \approx 2961 \text{ mm}^2$.

Answers to Exercises 8-7-c

1. (a) 1,000,000 (c) $\frac{1}{1,000}$ (e) $\frac{1}{1,000,000,000}$

   (b) 1,000,000,000 (d) $\frac{1}{1,000,000}$

2. The volume of the interior is $30 \text{ cm}^2 \times 6 \text{ cm} = 180 \text{ cm}^3$.

3. The volume of the interior is $2961 \text{ mm}^2 \times 59 \text{ mm} = 174,699 \text{ mm}^3$.

Answers to Exercises 8-7-d

1. (a) 4 Kgm + 4 hgm + 4 dkgm + 4 gm + 4 dgm + 4 cgm + 4 mgm.

   (b) 5 hgm + 6 dkgm + 7 gm + 8 dgm.

2. (a) 30 c.c. of water weigh 30 gms.

   (b) 0.030 kgms.

Answers to Exercises 8-7-e

1. (a) 6 kl + 6 hl + 6 dkl + 6 l + 6 dl + 6 cl + 6 ml.

   (b) 2 l + 0 dl + 3 cl.

2. 1 cc = 1 ml $\because$ 30 cc = 30 ml.

3. 1 ml = 0.001 l $\because$ 30 ml = 0.030 l.

Answers to Exercises 8-7-f

1. (a) 2.54 cm = 1 in. $10 \times 2.54 = 25.4$

   $\because$ 25.4 cm = 10×1 in = 10 in.

   (b) 1 meter $\approx$ 1.1 yds. $\because$ 100 meters $\approx 100 \times (1.1) \approx$ 110 yds.

   (c) 200 meters $\approx$ 220 yds.

   (d) 400 meters $\approx$ 440 yds.

   (e) 800 meters $\approx$ 880 yds.

   (f) 1500 meters $\approx$ 1100 yds + 550 yds $\approx$ 1650 yds.

   (g) 1 kilometer $\approx$ .6 miles, $\because$ 10 km $\approx$ 6 miles.

   (h) 100 km $\approx$ 60 miles.

(i) 1 kgm 2.2 lbs. 10 kgms 22 lbs.

(j) 100 kgm 220 lbs.

(k) 4 liters 4 quarts 1 gal.

(l) 60 liters 15 × 4 qts. 60 - 4 gals 15 gals.

(m) 1 metric ton 1.1 tons; 10 metric tons 11 tons = 11 × 2000 = 22,000 lbs.

(n) 100 metric tons 110 tons 220,000 lbs.

2. No allowance is made for the approximate nature of measures

(a) 531,441 cu. in. 307.5 cu. ft. 19188 lb.

(b) 8.869743 m^3. 8869.743 ℓ. 8869.743 kg.

(c) One set of computations needs to be made for the metric answers, so (b) should take less time. One more measure (liters) is found. Gallons for part (a) would be equivalent. 531,441 cu. in. 2300.6 gal.

*3. Gold is weighed in Trey units, while feathers are weighed in Avoirdupois units. The pound of feathers weigh more.

8-8. Suggested Test Items

True--False

1. $\frac{8}{25}$ is another name for the number 32%. True.

2. An increase in the price of an item from $20 to $28 is an increase of 40%. True.

3. If a class has a total of 32 pupils, 20 of them boys, the number of boys is 60% of the numbers of pupils in the class. False.

4. Five percent of $150 is the same amount of money as 7.5% of $100. True.

5. 62.5% and $\frac{5}{8}$ are names for two different numbers. False.

Multiple choice

1. Six percent of $350 is

(a) $210.00 (b) $21.00 (c) $2.10 (d) $2100.

(e) none of these. b

2. If 8% of the number 5400 is computed, the correct answer is

(a) More than 30 but less than 90

(b) More than 3 but less than 5

(c) More than 45 but less than 450

(d) More than 500 but less than 1000

(e) None of these c

3. If $\frac{1}{2}$% of $320.00 is computed, the answer is

(a) $16 (b) $160 (c) $3.20 (d) $1.60

(e) None of these d

4. In a class of 42 pupils there are 25 boys. The number of boys is what percent (nearest whole percent) of the number of pupils?

(a) 60% (b) 59% (c) 58% (d) 61%

(e) None of these a

5. In the class of 42 pupils there are 17 girls. The number of girls is what percent (rounded to the nearest tenth of a percent) of the number of pupils?

(a) 40.4% (b) 40.6% (c) 39.9% (d) 40.7%

(e) None of these e

6. Which of the following terminate when written as decimals?

(a) $\frac{1}{3}$ (b) $\frac{5}{13}$ (c) $\frac{2}{9}$ (d) $\frac{9}{8}$ (e) $\frac{11}{6}$ d

7. Select the best answer to finish the sentence: When a rational number is changed to a decimal

(a) it will be a terminating decimal

(b) it will be an exacr number

(c) it will be a repeating decimal

(d) it will be an infinite decimal

(e) one cannot tell what kind of a decimal it will be

c

Matching

Choose items from Column B which make the statements in Column A correct. Place the letters of your choices from Column B in the spaces to the left in Column A.

Column A

b 1. The digit 9 occupies the _____ decimal place in the decimal numeral 3284.569

d 2. If a and b are two numbers and b $\neq$ 0, the _____ of a to b is the quotient $\frac{a}{b}$

k 3. 1000 cm^3 of water weighs _____ kgms.

a 4. In the decimal numeral 9384.562 the digit 9 occupies the _____ decimal place.

h 5. The decimal numeral 473.45 rounded to the nearest tenth is _____.

Column B

(a) thousand

(b) thousandth

(c) percent

(d) ratio

(e) constant of proportionality

(f) 473.5

(g) 1000 kilograms

(h) 473.4

(i) terminating

(j) repeating but not terminating

(k) 1 kilogram

(l) 25.4

(m) 254

__i__ 6. The decimal numeral for the rational number $\frac{1}{8}$ is a_____ decimal.

__l__ 7. 2.54 cm. equals_____millimeters.

__c__ 8. The word_____is used when a ratio is expressed with a denominator of 100.

__j__ 9. The decimal numberal for the rational number $\frac{2}{7}$ is a_____ decimal.

__e__ 10. If two quantities are related so that the ratio of corresponding values of one quantity to the other is always the same, then the value of this ratio is called the_____.

Verbal Problems

1. What commission will a real estate agent receive for selling a house for $15,400 if his rate of commission is 5%? $770
2. The sale price on a dress is $22.80 and the marked price showing on the price tag is $30.00. What was the rate of discount? 24%
3. An increase in rent of 5% of the present rent will add $3.50 to the monthly rent that Mr. Johnson will pay. What is the monthly rent that Mr. Johnson now pays? $70

4. A family budget allows 30% of the family income for food. If the monthly income of the family is \$423, what amount of money is allowed for food for the month? \$126.90

5. Dorothy is 5 feet tall (to the nearest inch) when school opened in September. In June her height was 5 feet 3 inches. What is the percent of increase in her height? 5%

6. (a) Write $4(1) + 3\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^3}\right)$ in positional notation

4.305

(b) Write 0.12 in expanded form $1\left(\frac{1}{10}\right) + 2\left(\frac{1}{10^2}\right)$

7. Find the decimal to the nearest tenth for $\frac{17}{38}$. 0.4

8. (a) Round 14.657 to the nearest tenth. 14.7

(b) Round 5.9349 to the nearest hundredth. 5.93

9. Find the decimal to the nearest hundredth for $\frac{21}{13}$. 1.62

10. Show that $\frac{17}{16}$ is terminating when written as a decimal.

1.0625

11. Look at the computation of the decimal for $\frac{2}{7}$ in Section 3. If the division were continued 14 places, what would you predict the 14th digit to be? 8

12. At a certain time of the day a man 6 feet tall casts a shadow 8 feet long. At the same time a tree casts a shadow of 40 feet. How tall is the tree? 30 feet

13. The greatest possible error in a measurement that is made to the nearest centimeter is how much? one-half centimeter or 5 mm.

14. How much is the difference of the sum of 1.05 plus 0.75 and the sum of 0.5 plus 0.125? 1.175

15. The decimal numeral for the rational number $\frac{1}{11}$ is _____.

$0.0909\overline{09}$

16. The rational number for the decimal numeral $0.4545\overline{45}$ is_____.

$\frac{5}{11}$

17. Calculate the measure (in cm) of the perimeter of a square
whose edges measure 30 cm. 120 cm.

18. Calculate the area (in cm^2) of the interior of a square whose
edges measure 30 cm. 900 cm^2.

19. Calculate the volume (in cm^3) if the interior of a cube whose
edges measure 30 cm. 27,000 cm^3.

20. Calculate the capacity in kiloliters of a cube whose edges
measure 30 cm. 0.027 kl.

21. Calculate the weight of water in kilograms contained in a cube
whose edges measure 30 cm. 27 kgm.

CHAPTER 9

PARALLELS, TRIANGLES, PARALLELOGRAMS, RIGHT PRISMS

"Informal geometry" as presented in this chapter, is concerned with the discovery of geometric principles through experimentation and, where feasible, the verification of empirical conclusions by deductive reasoning. Students perform as scientists in collecting data and then perform as mathematicians in the analysis and interpretation of the data they have obtained. Data needed to formulate a statement of a geometric property are obtained by measurement, with protractor or ruler, or by superimposing one figure on another.

Pupils are introduced to the use of deductive reasoning as a method for ascertaining what is true about a geometric figure, arguing from previously stated principles and definitions. We reserve for a later time the systematic organization of geometry as a deductive system, starting with postulates and undefined terms, and developing theorems and definitions on this basis.

The specific purposes for which this chapter were planned are these:

1. To develop awareness of the occurence of points, lines, and planes and their intersections.

2. To introduce certain geometric concepts and relations as listed below.

3. To give the pupils experience in verification of experimental results by informal deductive argument on the basis of previously stated principles.

The major topics are:

1. Some angle relationships in the figure formed by parallel lines and their transversal.
2. The angle and side relationships in a triangle.
3. The angle and side relationships in a parallelogram.
4. Areas of parallelograms and triangles.
5. Definition and volume of a right prism.

Some General Observations and Suggestions.

As in other chapters, precise terminology is emphasized throughout the text material. It is necessary to make this emphasis because many of the words that are casually used by seventh graders are not as clearly understood by their users as we hope. At this level, the consequences of casual language are not always serious but may become so as students proceed in their mathematical studies. All of the terminology developed in previous chapters should be used whenever such usage clarifies and simplifies geometric statements but care should be exercised that in our attempts to be as exact as possible, we do not make a complicated thing out of what may be, essentially, a very simple idea. To avoid this situation, it is suggested that meanings be given first in words of common usage and then in the more precise terminology. The translation from common usage to precise usage then becomes an exercise in analytical thinking.

Ideally, once a word that is commonly used is pre-empted for a special meaning in a new vocabulary the new meaning must be adhered to from that point on. In actual fact, however, it is often difficult to convince a seventh grader that he should do this. In this case we should accept, for the time being, his way of speaking, evaluate and discuss what ideas he is attempting to present, and then encourage him to rephrase his statements according to the more precise wordage.

Since students often learn best by imitation and habit formation, it is suggested that the teacher become thoroughly familiar with the new terminology and use it at every possible occasion. Through the simultaneous use of both the common and the precise ways of speaking it is hoped the student will become more and more proficient in the latter and grow to appreciate its value until he eventually uses it as a matter of course.

Some particular terminology used in the text may need further clarification. "Intersect in the empty set" and "are parallel" as applied to lines have the same meaning and may be used interchangeably where the lines are understood to be in the same plane. "Intersect in the empty set" or "are skew" have the same meaning when applied to lines that are not in the same plane. The front edge of the ceiling and the side edge of the floor are skew and have no point of intersection. Which of these phrases to use in a particular context depends on the conclusions one wishes to draw. If the question is, "What are all the possible intersections of two lines in space?", one of the possible intersections is "the empty set". This is a phrase that might seem preferable to "non-parallel lines have no intersection". On the other hand, if the question is "How are the opposite sides of a parallelogram related?", then "they are parallel" might seem preferable to "they intersect in the empty set". Use your own judgment in matters of this kind. It may be noted that the authors at times say "the lines do not intersect" even though this is not strictly so in the precise set terminology. We say "the lines may intersect in the empty set". Since the phrase "do not intersect" is so commonly used, it seems desirable to use it here. It is also thought that there is some advantage in presenting some ideas in more general terms, which set language permits.

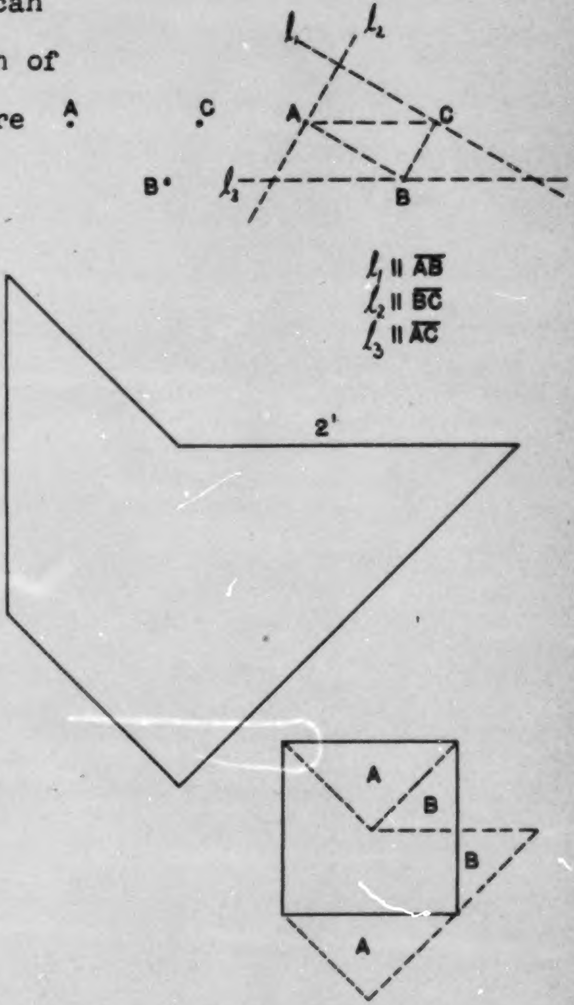
The chapter includes a few deductive developments of a more or less informal nature. One of the problems arising in such a development is that pupils usually fail to appreciate the need for justifying statements with reasons previously adjudged acceptable to the group as a whole. One proposal that might impress them with the fact that only previously stated and accepted properties, definitions, and reasons should be used is to suggest that football and basketball games would be much more interesting if in each game the rules were changed without consulting anybody and that new rules be made up as the game goes along! It might be an interesting game but hardly a fair one! An occasional reminder about "making up rules as you go along" is usually sufficient to make the point desired.

Frequently students are asked to make a general statement about a property or the results obtained through experiment. In the text such statements are partially written so that the grammatical form of the statement is suggested without hinting too strongly at the mathematical ideas involved. Before considering these, students should be encouraged to formulate their own statements of principles and properties but such statements should be very closely examined to ensure that the meaning is precise and clear. When a statement seems satisfactory to all, then show pupils the formulation in the answer section for comparison. They then may use these statements as models in future work in this chapter.

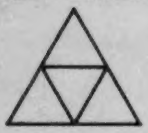
About a week prior to the introduction of this chapter, pictures and articles pertaining to plane geometry could be placed on the classroom bulletin board to arouse curiosity and supplement the historical facts briefly mentioned in the first section. Here are two puzzles that will add interest to the display:

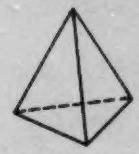
1. Drawing a triangle is easy but can you draw a triangle so that each of the dots lettered A, B, and C are midpoints of its sides?

2. A very thrifty cabinet-maker wished to construct a table top two feet square out of a piece of plywood shaped as in the figure. He was able to do this with only two sawings. If you are as clever as the cabinet-maker, you can do the same.


A
C
B
l_1
l_2
A
C
l_3
B
$l_1 \parallel \overline{AB}$
$l_2 \parallel \overline{BC}$
$l_3 \parallel \overline{AC}$
2'
A
B
B
A

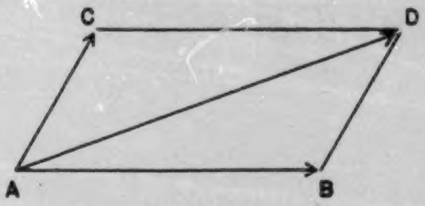
Pictures and geometric diagrams suitable for posting are not always easy to find but some good ones can be obtained from such magazines as "Scientific American", "Fortune", "Popular Science", "Popular Mechanics", "Life", and, of course, some technical magazines. A pattern of "oak tag" paper might be posted along with a completed model for the construction of a regular tetrahedron as an activity in equilateral triangles.





The following hints are given about applications for the purpose of motivation. These may be supplemented by material found in the introductory chapters of plane geometry texts that are available.

There is never an over-supply of good thinkers. The world needs people who can begin with a body of facts, relate them, and think through to logical conclusions. The study of geometry helps to train such people. In the aircraft industry there is a great demand for workers trained in geometry because there is a considerable amount of geometric knowledge involved in the construction of an airplane. The main problem is to find out how air will flow about an airplane of given shape moving in a given direction at a given speed. From this the lifting force and the air resistance may be calculated. The parallelogram of forces may be used for an illustration. In order to find the single force equivalent to two forces acting simultaneously at a point we can draw a diagram like this in which the given forces are represented in magnitude and direction by the segments $\overline{AB}$ and $\overline{AC}$. We complete the parallelogram, and the diagonal AD gives the magnitude and direction of the resultant force.



Geometry is also used to figure out the forces in an electromagnetic field, and why rubber is elastic, and how an oil company should schedule its production. In the theory of relativity and in the design of agricultural experiments completely different concepts of space are used. Today, the physicist, the chemist, the biologist, the engineer, the economist, the psychologist, and the military strategist use geometry in ways far removed from surveying, some of which were not even discussed or dreamed of only fifteen years ago.

9-1. Vertical and Adjacent Angles

Concepts to be developed:

1. When the intersection of two lines is not empty, four angles and four half-planes are determined.
2. Two angles which have a common ray and whose interiors have no points in common are called adjacent angles.
3. Non-adjacent angles determined by the intersection of two lines [in a single point] are called vertical angles.
4. When two lines intersect in a point, they determine four different half-planes. Two of these half-planes intersect in the interior of one angle in a pair of vertical angles. The interior of the other angle in the pair of vertical angles is the intersection of the remaining two half-planes.
5. The angles in a pair of vertical angles are equal in measure.

Suggestions.

Review very briefly the possible intersections of two lines in space using two meter sticks, pointers, or pieces of coat-hanger wire to represent a pair of lines intersecting in space. If the intersection is not the empty set, both wires should be grasped in one hand at their point of intersection so that the other hand is free to indicate parts of the figure. Use this same device to suggest adjacent angles and vertical angles. By this procedure, students will be encouraged to think of these ideas in terms of "general" space and not just that portion of space represented by the chalkboard or paper. Also, it is often more convenient and more time-saving to carry a geometric figure to the students in this way than to have the students carry themselves to the same figure drawn on the chalkboard.

Pupils may say "Angle A is a vertical angle". This statement is no more correct than "a parallel line" or "a trouser" or "a scissor". In fact, if the first statement makes sense, then we could refer to any angle formed as being a vertical angle. "Vertical angle" would then have no meaning. Instead of "Angle A is an adjacent angle" we should insist that "adjacent angle" be used only in a phrase which includes some mention of a pair of angles.

In Problem 5(e) it is not necessary to formalize the subtraction axiom. The notions involved in the deduction required should already be understood and accepted by the group from previous experience with number sentences.

In part (e) of Problem 6, the reason for $m(\angle x) = m(\angle z)$ should be developed as a statement of the equality of two numbers and that this particular equality is true because $m(\angle x)$ and $m(\angle z)$ are "two names for the same number."

Answers to Questions in Section 9-1

Yes. The plane which is a side of BC and contains point D intersects the plane which is a side of ED and contains point C to form the interior of $\angle$ CAD.

Answers to Exercises 9-1

1. (a) $\angle BAD$, $\angle EAC$

   (b) $\angle BAD$, $\angle EAC$

   (c) $\angle EAB$, $\angle CAD$.

2. (a) $\angle CAD$

   (b) $\angle BAD$ and $\angle EAC$

   (c) Two pairs of vertical angles are formed.

3. (a) $40 = m(\angle BAE)$ $40 = m(\angle DAC)$

   (b) $140 = m(\angle CAE)$ $140 = m(\angle BAD)$

   (c) The angles of a pair of vertical angles seem to have equal measures.

   (d) They do appear equal.

4. Property 1: When two lines intersect, the two angles in each pair of vertical angles which are formed have equal measure.

*5. (a) 180

   (b) 180

   (c) 120, 120

   (d) 110, 110

   (e) $180 =$ sum of $m(\angle BAE)$ and $m(\angle BAD)$

   $180 =$ sum of $m(\angle BAE)$ and $m(\angle CAE)$

   Then $m(\angle BAD) = m(\angle CAE)$ since they both must be names for the same number.

6. (a) 180

   (b) 180

   (c) Subtract $m(\angle y)$ from 180

   (d) $m(\angle x) = 180 - m(\angle y)$;

   $m(\angle z) = 180 - m(\angle y)$

   (e) $m(\angle x) = m(z)$

   (f) Yes

7. (a) Yes, two lines which are skew.

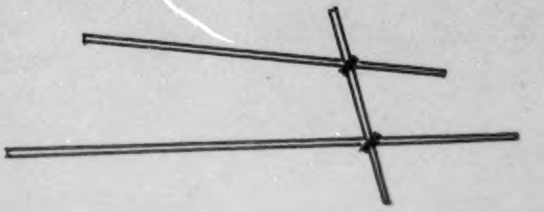
   (b) The edge of the floor at the front of the room and the edge of the ceiling at one side.

9-2. Three Lines in Space

Concepts to be developed:

1. A line which intersects two or more lines in distinct points is called a transversal.
2. In any figure consisting of lines whose intersections are not empty, there are at least two pairs of vertical angles and at least four pairs of adjacent angles.
3. Two angles whose measures add up to 180 degrees are called supplementary angles.
4. Supplementary angles are not necessarily adjacent.
5. Two different angles whose interiors lie on the same side of a transversal such that a complete ray of one is contained in a ray of the other, are called corresponding angles.

It will prove helpful to make a demonstration model constructed of three flat sticks free to turn on pivoted connections made with rubber bands, small round-head bolts, or of clinched tacks or nails.



On the chalkboard represent parallel lines in other than horizontal positions at first. Otherwise, students get the habit of thinking of parallel lines as being horizontal and have difficulty seeing them in complicated figures where the parallel lines are not horizontal. Colored chalk lends interest to a chalkboard drawing and is effective in calling special attention to certain portions of a figure.

Pupils should draw figures according to the directions in the exercises even though the figures appear in the text. This activity helps to fix the features of a drawing in their minds. Students' drawings frequently are too small. In the exercises of this section lines should be represented by segments about two to three inches long. Free-hand sketches are acceptable if done carefully.

Problem 6 of the exercises is of great importance and by no means should be omitted. The basic definition included in this problem requires that it be done by all pupils.

Answers to Questions in Section 9-2

Yes; yes; no.

Answers to Exercises 9-2

1. (a) Four pairs (b) Eight pairs (c) Yes
2. (a) Triangle R Q S (b) Six pairs (c) Twelve pairs
3. (a) Line t is the transversal and intersects ℓ_1 and ℓ_2
   (b) Eight angles
   (c) Four pairs
   (d) The angles in any pair of vertical angles have equal measures.
   (e) Yes, because $\angle h$ and $\angle f$ are vertical angles.
   (f) No, adjacent angles may or may not have equal measures.
4. (a) Eight pairs (b) Sum of measures is 180
5. (a) Yes
   (b) No
   (c) Yes (Neither straight nor reflex angle has been defined.)
   (d) $m(\angle g) = 100$ $m(\angle e) = 100$ $m(\angle f) = 80$
   (e) Yes (f) Yes

6. (a) ∠ a and ∠ g
   (b) Yes
   (c) No
   (d) Four pairs
   (e) No
   (f) The measure of each angle is 90.
   (g) Yes

*7. (a) Each shows two lines with their transversal.
(There are others)

(b) The figures differ in the number of pairs of lines whose intersection consists of one point. (There are others.)

(c) All three lines may be parallel.

(d) The union of the intersections of each two lines of a group of three different lines in the same plane consists of __0__ , __1__ , __2__ , or __3__ points.

*8. With three planes in space the cases are:

(a) All pass through a point. (The corner where two walls and the ceiling meet.)

(b) Each two planes intersecting but their lines of intersection are parallel. (In one type of house, the planes of the roof and the plane of the attic floor.)

(c) Two planes parallel but both intersected by the third (floor, ceiling and one wall.)

(d) Three parallel planes (a stack of shelves.)

9-3. Parallel Lines and Corresponding Angles

Concepts to be developed:

1. When a transversal intersects two lines in the same plane and the corresponding angles have unequal measures, then the two lines will intersect.

2. When a transversal intersects two lines in the same plane and the corresponding angles have equal measures, then the two lines do not intersect.

Suggestions.

The emphasis should be on the children discovering the relationships by their own observations. Be careful not to "kill" their interest by telling them the answers. Ask leading questions and try to draw the answers our of the students.

If class time is short, the experiments in this section may be speeded up if each student is assigned the measuring for two cases and then reports his findings for tabulation on the chalkboard. It might be well to carry out other experiments in this same manner.

If an overhead projector is available, by all means use it for this section. On one piece of plastic film, draw with china-marking pencil line r_2 and transversal t. On another piece draw line r_1. By superimposing the second piece on the first and projecting the image on the chalkboard, line r_1 can be rotated to different positions through Point A and observations as to angle measures and intersections be tabulated on the chalkboard. Since the figure is projected on the chalkboard, marks may be erased without obliterating the figure.

Answers to Class Exercises and Discussion 9-3

1. Yes, they intersect on the left side of t.
2. Yes, r_1 and r_2 intersect on the right side of t.
3. Third column, third entry is "the empty set". If the corresponding angles are unequal, r_1 and r_2 intersect in a point to the right or left of the transversal t. If the corre-

sponding angles are equal, r_1 and r_2 are parallel.

4.

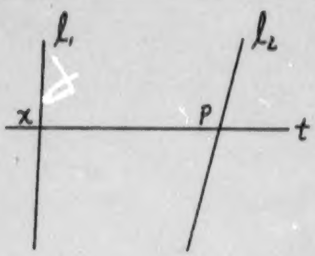
Intersection of r_1 and r_2
right side
empty set
left side

5. measures , intersect

6. plane , parallel

Answers to Exercises 9-3

1.



Intersection of $_1$ and $_2$
(a) below
(b) parallel
(c) above
(d) below
(e) parallel
(f) above
(g) below
(h) parallel
(i) above

2. As in classroom (opposite edges of ceiling, etc.)
3. 90. Minimum distance should occur when the transversal is perpendicular to the lines. (This idea is more fully developed in Section 9-7.)

9-4. Converses (Turning a Statement Around)

Concepts to be developed:

1. A converse of an "if--then" sentence may be formed by reversing the order of the "if" clause and the "then" clause.
2. A converse of a statement may be true or false regardless of whether the original statement is true or false.

Suggestions.

Caution students that the meanings of words used in the sample statements of the text are understood to be those in the widest use. "Mary and Sue" are not names for ships!

Care must be taken not to refer to a converse as *the* converse because for any one statement there may be several different converses. To form a converse it is only necessary to exchange one fact of the "if" clause with one fact of the "then" clause and where these clauses contain more than one fact, there will be several converses. It is felt that the "turning around" device is as far as one needs to go for seventh graders.

It might prove helpful to clarify the meaning of "true" or "false" before tackling the exercises. Without going into details of logic, it should be sufficient to propose that a "false" statement is one that is not always true and a statement is not considered true if at least one counter-example can be found. For instance, let us examine the statement: "The set of whole numbers is closed under subtraction". As a counter-example one might say, "There is no whole number which added to five gives three and, therefore, (3 - 5) is not a name for a whole number". This one counter-example is all that is needed to deny the quotation. Emphasize that only one counter-example is needed to prove a statement false.

While one counter-example can be used to show that a statement is false, it is a great deal more difficult to show that a statement is true. In this section we do not expect that the students or teacher will prove that statements are true. In Section 6 a geometric proof is given for the property of the sum of the angles of a triangle, and it is suggested in the exercises that students try to discover proofs for several other properties. In this section only inductive arguments can be given to indicate that a statement is true. Students should be reminded that even though thousands of cases were investigated and all were in agreement with the statement or property, still the property would not necessarily be true. Also, failure to find a counter-example will not constitute a proof, since it is always conceivable that someone else might find such an example.

In "if--then" statements, the "if" clause is considered to be the postulate or postulates. In mathematics it is recognized that even a formal proof is based on postulates and that the property to be proved is true only if the postulates used in the proof are true. Mathematicians do not recognize any absolute truth in mathematics. A mathematical agreement, based on postulates which may not necessarily even be in agreement with experience, is considered a proof, provided deduction is used correctly. Again, it is recognized that the validity of the proof does not determine the truth of the property. The truth of a property is completely dependent on the truth of the postulates.

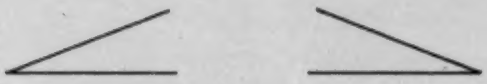
Answers to Questions in Section 9-4

(b) Not always

1. Yes ; No
2. Yes ; Yes
3. Statement "a" is true.
   Statement "b" is false.

Answers to Exercises 9-4

1. One of many examples is:



No.

2. (a) false (d) true
   (b) true (e) true (if no amputees)
   (c) true (f) true

3. (a) If Blackie is a cocker-spaniel, then Blackie is a dog. (true)
   (b) If we cannot see the sun, then it is night. (false)
   (c) If it is a holiday in the U.S, then it is July 4th. (false)
   (d) If Robert is the tallest boy in his class, then he is the tallest boy in his school. (false)
   (e) If an animal has four legs, then the animal is a horse. (false)
   (f) If an animal has thick fur, then the animal is a bear. (false)

4. (a) true (c) false (e) false
   (b) true (d) true (f) true

5. (a) If two lines are parallel and are intersected by a transversal, a pair of corresponding angles have equal measures. (true)
   (b) The angles are equal in measure. Classmates should have similar results.
   (c) True

6. If a transversal intersects two lines in a plane and the two lines intersect in a point, then the angles in a pair of corresponding angles formed are unequal in measure. (true)

7. Yes. If a figure is a simple closed curve, then it is a circle. (false)

   If a figure is a circle, then it is a simple closed curve. (true)

9-5. Triangles

Concepts to be developed:

1. A set is determined in accordance with a common property.

2. There are three sets of triangles determined according to the measures of their sides.
   (a) The set of isosceles triangles has as members triangles which have two sides that are equal in length.
   (b) The set of scalene triangles includes triangles which have no two sides with the same measure.
   (c) The set of equilateral triangles includes triangles which have three sides equal in length.

3. An angle and a side of a triangle are said to be opposite each other if their intersection contains just the endpoints of the segment referred to as side.

4. If two sides of a triangle are equal in length, the angles opposite these sides have equal measures.

5. If two angles of a triangle have equal measures, then the sides opposite these angles are equal in length.

Suggestions.

Remind students that a set should be "well-defined". There must be no doubt or question as to what is or is not a member of a set. This does not mean, however, that a definition of set should be attempted. The set of beautiful paintings in the world is not a "well-defined" set because what is beautiful is controversial. The set of triangles is not a "well-defined" set if it is not clear whether spherical triangles are to be included. In our text, since all triangles are understood to be plane figures the set of triangles is clearly defined. In other words a set is well-defined if and only if there is no doubt as to whether a thing belongs in the set or not. This idea is important in defining the set of isosceles triangles and emphasizes the need for precise and complete terminology.

Students should try to answer questions in the text as they reach them and not read ahead for the answers. Answers are included in the text so that, if desired, all or a portion of the text may be assigned for reading outside of class.

Have a number of soda straws measured and creased before class begins. This exercise may seem easy to most pupils but even the brighter ones will jump to incorrect conclusions about a straw divided into three pieces of 2", 3", and 5". Soda-straw figures give the class an opportunity to handle triangles in regions of space other than the chalkboard or the drawing paper. Make certain that the soda-straw figures are understood to be merely representations of triangles and not actual triangles.

Answers to Questions in Text 9-5

(a) scalene	(c) isosceles	(e) equilateral
(b) equilateral	(d) isosceles	

Yes, $\angle B$ is opposite $\overline{AC}$ because only the end points of $\overline{AC}$ are points in $\angle B$.

No, because all points (not just the end points) of $\overline{AC}$ are points in $\angle C$.

$\angle C$ is opposite $\overline{AB}$; $\angle A$ is opposite $\overline{BC}$. (Similar reason.)

$\triangle ABC$ is isosceles because two of its sides have equal measure.

The procedure suggests $m(\angle B) = m(\angle C)$

The same relationship of two angles is observed in other isosceles triangles.

Property 3. If two sides of a triangle are equal in length, then the angles opposite these sides have equal measures.

The sides opposite $\angle A$ and $\angle B$ in this triangle are equal in length.

The sides opposite $\angle A$ and $\angle B$ in each triangle drawn are equal in length.

The triangles are isosceles triangles because two sides in each are of equal length.

Property. If two angles of a triangle are equal in measure, then the sides opposite these angles have the same length.

Yes, "converse".

Answers to Class Discussion Problems 9-5

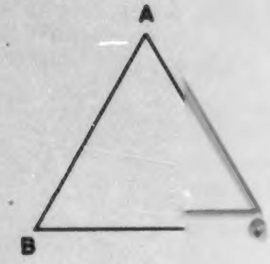
1. Yes. Advantages
   1. Useful when only angle measures are known.
   2. Ruler may not be available.
   3. Others.

Disadvantages 1. Protractors not as generally available as rulers.

2. Angle measure is more difficult to estimate for many people than linear measure.

3. Others.

2.



1.	$m(\angle A) = m(\angle B)$	1.	Given
2.	$m(\overline{BC}) = m(\overline{AC})$	2.	Converse of Problem 3
3.	$m(\angle C) = m(\angle B)$	3.	Given
4.	$m(\overline{AB}) = m(\overline{AC})$	4.	Converse of Property 3
5.	$m(\overline{BC}) = m(\overline{AC}) = (\overline{AB})$	5.	All names for the same number.

Answers to Questions in Section 9-5

Figures (b) and (f) are triangles.

Common property: At least two sides are equal.

Common property: All three sides have equal measures.

Common property: No two sides have the same measures.

No; No

Soda-straw figures.

(a) scalene	(d) isosceles
(b) equilateral and isosceles	(e) equilateral and isosceles
(c) isosceles	

Triangle ABC (first drawing)

Yes, because A and C are the only points shared by angle B and side $\overline{AC}$.

No, because points A and C are not the only points shared by angle C and $\overline{AC}$.

Angle C and side $\overline{AB}$, because points A and B are the only points shared by angle C and side $\overline{AB}$.

Triangle ABC (second drawing)

Isosceles, because at least two sides have equal measures.

They are equal.

The measures of at least two angles in each triangle are equal.

Property 3: *If two sides of a triangle are equal in length, then the angles opposite these sides have equal measures.*

Triangle ABE (third drawing)

They are equal. The sides opposite the equal angles are equal.

Isosceles triangles.

Property: If two angles of a triangle are equal in measure, then the sides opposite these angles are equal in length.

Yes. They are converses.

1. Yes.

2. By definition it is sufficient to show that two sides of a triangle are equal in measure in order to establish that it is isosceles. This can be done by taking any two of the three equal angles and using the converse of Property 3 to prove that the opposite sides are equal.

Answers to Exercises 9-5

4. Not necessarily, because an equilateral triangle must have all three sides equal and an isosceles triangle does not satisfy this requirement.

5. Yes, because an equilateral triangle does satisfy the requirements for being isosceles by having two sides of the same length. "Only two sides equal in length" is not part of the requirement for isosceles triangle.

6. P opposite $\overline{QR}$
   Q opposite $\overline{PR}$
   R opposite $\overline{PQ}$

7. (a) No. To make a triangle the first two sections would have to make an angle less than 180 in measure. But the first two sections, even if held end to end in a straight line, would precisely reach from one end of the 3-inch section to the other.

   (b) No. If the end pieces were folded toward each other, they would not meet.

   (c) The sum of the measures of any two sides of a triangle must be greater than the measure of the third side.

9-6. Angles of a Triangle

Concepts to be developed:

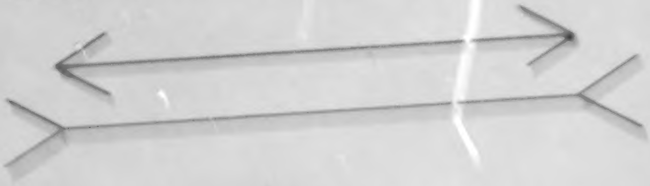
1. The sum of the measures of the angles of a triangle in a plane is 180.
2. A property may be suggested by inductive reasoning but is proved by deductive reasoning.
3. The proof of Property 4 applies to any triangle in a plane.

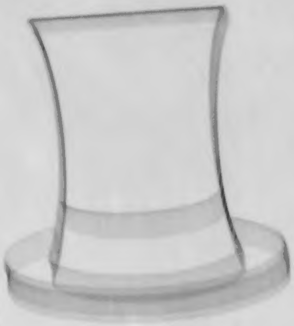
Suggestions:

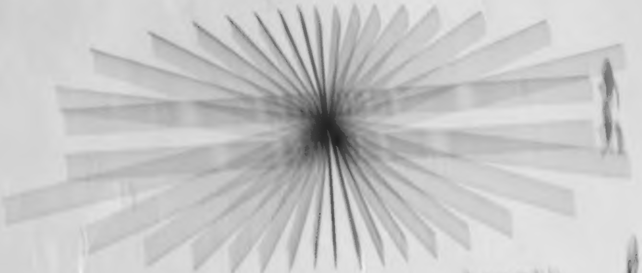
Since the class exercises require the folding and cutting of triangular shapes, much class time may be saved by supplying the class with a quantity of such shapes already cut out instead of spending time in the distribution and collecting of scissors. By using a paper cutter and cutting several pieces of paper at a time, a sufficient supply can be easily obtained. Triangles should be fairly large and as different as possible.

Whether the mounting and pasting suggested in the text is done in class or not is optional. In fact, this particular activity could be a good "lesson clincher" if made a part of the homework assignment. However, the tearing and folding for the purposes of experiment should be done in class as suggested.

The angle sum for a triangle is first developed experimentally and then proved deductively. To some pupils it may seem silly to prove deductively what is perfectly obvious intuitively. One way to meet this criticism is to shake these students' faith in the infallibility of their intuition. Here are a few optical illusions, quickly and easily drawn on the chalkboard, that might do the trick:







AB and CD have the same length. ℓ_1 and ℓ_2 do not bulge apart. The top hat is as wide as it is tall.

In this paragraph are mentioned other ways helpful in emphasizing the need for verification of "obvious" or intuitive conclusions. Test students ability to distinguish between two sharp points by touch alone (try this on finger tips first and then on the back of the neck), or ask what the bill would be for shoeing a horse at one penny for the first nail, two for the second, four for the third, eight for the fourth, etc. for four shoes with eight nails to the shoe. Tell about the native of China who for seventy years, upon awakening in the morning, noted that the first person he saw was another Chinese. On the eve of his seventy-first birthday, after having made the same observation approximately 25,915 times, he went to sleep certain in the knowledge that the first person he would see in the morning would be another Chinese. It was a Russian!

Spend some time with the class discussing why the deductive proof applies to all planar triangles and why the conclusions drawn from empirical data in exercises 1 and 2 apply only to the particular triangles tested.

It is recognized that Problem 4 of the classroom exercises will be difficult, particularly the attempt to decide on a legitimate reason for each step. This is the only example of a formal deductive proof given in the chapter, however, and it is hoped that the teacher will go through it with the class, though not expecting real mastery. It might be helpful to recall for the class the idea of "making up the rules after the game has started" as suggested in the second paragraph, Page 4 of this commentary.

Answers to Questions in Section 9-6

1. (a) $m(\angle DAC) = 180$.
   (b) The sum of the measures of $\angle 1$, $\angle 2$, and $\angle BAC$ is 180.
2. Point C seems to fall on Point G, also.
   Point B seems to fall on Point G, also.
   (a) $\overline{DB}$ lies on $\overline{DG}$, $\overline{EB}$ lies on $\overline{EG}$
   (b) 180 (c) Yes (d) Yes

3. Property 4. The sum of the measures in degrees of the angles of any triangle is 180.

4. (a) Yes, because of Property 2a.
   (b) Corresponding angles. Yes. Converse of Property 2a.
   (c) Vertical angles. Yes. Property 1.
   (d) Were drawn so as to have equal measures.
   (e) The measures in the sum on the left are equal to the measures in the sum on the right.
   (f) By definition of "sum".
   (g) Angles on one side of a line.
   (h) Two names for the same number as indicated in steps e and g.
   (i) The translation of the symbols in step h into words.

Answers to Exercises 9-6

1. 60

2. (a) 60 (b) 30 (c) 5 (d) 20

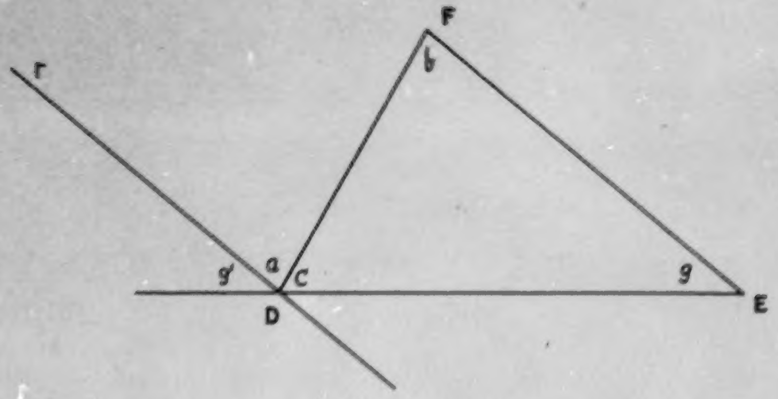
3. Either 65 and 65, or 50 and 80.

4. $m(\angle ABC) = m(\angle DEF)$ Property 4.

5. (a) 80 Property 4
   (b) 91 Property 4
   (c) 40 Property 3
   (d) 60 Property 3 and 4
   (e) 4 Property 4 and converse Property 3.

6. (a) Yes, Property 1
   (b) Yes, Property 2 or converse of Property 2a
   (c) 1. $m(\angle y) = m(\angle n)$ 1. Property 1
       2. $m(\angle y) = m(\angle u)$ 2. Property 2 or converse of Property 2a
       3. $m(\angle n) = m(\angle u)$ 3. Two names for the same number, $m(\angle y)$.

7.



Assign letters to other angles of the figure than these to which letters have already been assigned.

1. $m(\angle g') = m(\angle g)$ 1. As drawn

2. line r is parallel to FE 2. Property 2a

3. $m(\angle b) = m(\angle a)$ 3. Property of Problem 6

4. $m(\angle g') + m(\angle a) + m(\angle c) = 180$ 4. Angles on one side of a line.

5. $m(\angle g') + m(\angle a) + m(\angle c) = m(\angle g) + m(\angle b) + m\angle c)$

Each measure in the left hand sum is a measure in the right hand sum.

6. $m(\angle g) + m(\angle b) + m(\angle c) = 180$

Two names for the same number.

9-7. Parallelograms

Concepts to be developed:

1. The shortest segment from a point A to a line r lies on the line through point A perpendicular to line r. The length of this segment is called the distance from point A to line r.

2. If ℓ_1 and ℓ_2 are parallel lines, then the distance from a point on ℓ_1 to ℓ_2 is the same as the distance from any other

point on ℓ_1 to ℓ_2.

3. A line perpendicular to one of two parallel lines is perpencicular to the other also.

4. The distance between two parallel lines is the length of any segment of a line perpendicular to both given lines and having an endpoint on each.

5. A polygon is a simple closed curve which is the union of segments.

6. A quadrilateral is a polygon with four sides. A pentagon is a polygon with five sides.

7. Opposite sides of a quadrilateral are those which have no point in common. (They do not intersect.)

8. A parallelogram is a quadrilateral whose oppose sides are parallel.

9. The opposite sides of a parallelogram are equal in length.

10. The interior of a parallelogram may be separated by a diagonal into two triangular regions of the same size and shape.

Suggestions.

It is intended that as much as possible of the development be done in class. Note that the class material actually includes Problems 1 and 2 of the Exercises 9-7.

Because in developing Property 5 the drawing of parallelograms by the class would take up a large amount of time, it would be advisable to have prepared an ample supply of parallelogram cut-outs cut on the paper cutter. It would be possible to use these same cut-outs for Problem 5 of the exercises.

Answers to Questions in Text 9-7

m($\angle$TDA) = 140, m($\angle$TCA) = 90, m($\angle$TBA) = 30

$\overline{AC}$ is the shortest

No segments shorter than $\overline{AC}$

The shortest segment from a point A to a point of a line r is the segment from A which is perpendicular to r.

Yes. All are right angles because of Properties 2, 2a, 3 and 4.

Yes

The intersection of opposite sides of a quadrilateral is the empty set.

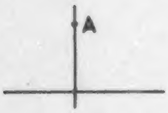
$\overline{AB}$ opposite $\overline{DC}$, $\overline{AD}$ opposite $\overline{BC}$

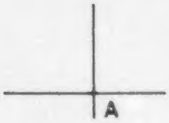
The opposite sides have equal length. Yes.

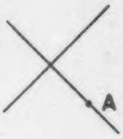
Yes, the area of the interior of one of the triangles is equal to one-half the area of the interior of the parallelogram because the interior of the triangle is one of two equal parts of the parallelogram.

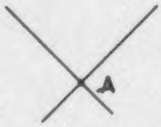
Answers to Exercises 9-7

1. As in classroom
2. As in classroom
3. (a) (d)
4.









5. The triangular pieces are equal in area. A diagonal separates the interiors of a parallelogram into two triangular regions which are equal in area and have the same shape.

6. Yes, lines perpendicular to one of two parallel lines are perpendicular to the other, also.

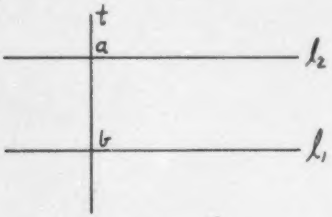
7. A J G F contains A J I D, A J H E, D I H E, D I G F, and E H G F. B M E K contains B K I L and I L M E.

8. On the order of the parallelograms listed in Number 7 the pairs of opposite sides which have the same measure are:

$[(\overline{AJ}, \overline{GE}), (\overline{JG}, \overline{AF})]$; $[(\overline{AJ}, \overline{ID}), (\overline{AD}, \overline{JI})]$;
$[(\overline{AJ}, \overline{EH}), (\overline{AE}, \overline{JH})]$; $[(\overline{DI}, \overline{EH}), (\overline{DE}, \overline{IH})]$;
$[(\overline{DF}, \overline{IG}), (\overline{ID}, \overline{FG})]$; $[(\overline{EH}, \overline{FG}), (\overline{EF}, \overline{HG})]$;
$[(\overline{KB}, \overline{EM}), (\overline{KE}, \overline{BM})]$; $[(\overline{KB}, \overline{LI}), (\overline{KI}, \overline{BL})]$;
$[(\overline{IL}, \overline{ME}), (\overline{IE}, \overline{LM})]$;

9. (a) $m(\overline{LM}) = 6$, $m(\overline{OM}) = 3$
(b) $m(\overline{KL}) = 12$, $m(\overline{OK}) = 4$
(c) In one pair of opposite sides the sides may be equal in length. In the remaining pair the sides must be unequal in length.

10. If a line is perpendicular to one of several parallel lines, then it is perpendicular to the others also.



1.	t is perpendicular to l_2 and intersects l_1	1.	As drawn
2.	$m(\angle a) = 90$	2.	Perpendiculars form right angles
3.	l_1 parallel l_2	3.	As drawn

4.	$m(\angle a) = m(\angle b)$	4.	Property 2a
5.	$m(\angle b) = 90$	5.	Two names for the same number
6.	Therefore, t is perpendicular to ℓ_1	6.	If two lines intersect to form a right angle, they are perpendicular.

9-8. Areas of Parallelograms and Triangles

Concepts to be developed:

1. Angles of a parallelogram at opposite vertices have the same measure.
2. Consecutive angles of a parallelogram are supplementary angles.
3. If one angle of a parallelogram is a right angle, the others must be also.
4. A rectangle is a parallelogram with one right angle.
5. The area of a parallelogram may be determined, given the length of a base and the length of the altitude to that base.
6. Any side of a parallelogram may be considered its base.
7. The interior of a triangle may be considered as half of the interior of a parallelogram whose base is the base of the triangle and whose altitude is the altitude of the triangle to that base.
8. Any side of a triangle may be considered as a base, and for each base there is a corresponding altitude.

Suggestions.

Chalkboard drawings of triangles and parallelograms should be so made that rarely do the altitudes discussed lie on a vertical line. Every effort should be made to have pupils realize that the altitudes of figures do not always extend in a vertical direction.

It will also be evident that bases do not necessarily lie on horizontal lines. Closely related to this is the fact that any one of three altitudes and their corresponding bases may be used to find the area of a scalene triangle and that there are, except for a rhombus or a square, two ways to calculate the area of a parallelogram.

Consult the answer section for comments on the development of exercises in the text. Particularly note that a parallelogram with one right angle is a rectangle. If time permits, this might be a good jumping off place for a discussion of what is "necessary and sufficient" in a definition and perhaps why in some statements the "if and only if" phraseology is used.

Answers to Questions in Section 9-8

Yes.

$(\overline{AB}, \overline{DC})$; $(\overline{AD}, \overline{BC})$

There are four cases.

$m(\angle A) + m(\angle B) = 180$

The angles of a parallelogram at two consecutive vertices are supplementary.

The measures of the angles of a parallelogram at two opposite vertices are equal.

If $\angle A$ is a right angle, all of the angles of the parallelogram are right angles. Yes, it is a rectangle. If $\angle A$ and $\angle B$ are not right angles and $\angle B$ is an acute angle, then $\angle A$ must be obtuse since $m(\angle A) + m(\angle B) = 180$.

Q' lies on $\overline{AB}$ extended because, as our figure shows, $\angle A$, which is an angle of $\triangle AQD$, has the same measure as the angle at B determined by $\overrightarrow{BC}$ and extension of $\overline{AB}$.

D Q Q' C is a parallelogram. $\angle D$ and $\angle DQQ'$ are consecutive angles and, therefore, supplementary. Since $m(\angle DQQ') = 90$, then $m(\angle D) = 90$.

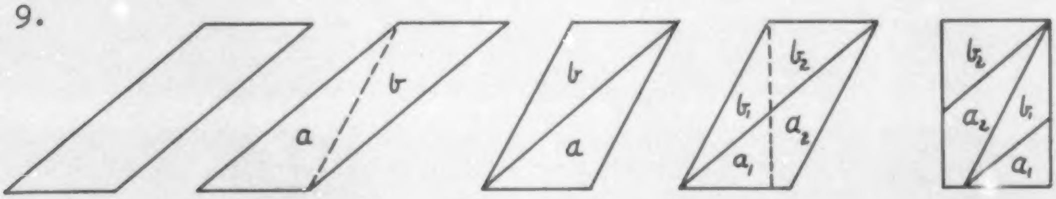
$m(\overline{AB}) = m(\overline{QQ'})$ because each represents the sum of measures respectively equal.

The number of square units of area in the interior of a parallelogram is the product of the number of linear units in the base and the number of linear units in the altitude to the base.

The number of square units in the area of the interior of triangle ABC is one-half the product of the number of linear units in the altitude to the base.

Answers to Exercises 9-8

1. (a) 12 sq. ft. (c) 80 sq. in. (e) 105 sq. m.
   (b) 6 sq. yd. (d) 72 sq. ft.
2. (a) 10 sq. in. (c) 28 sq. yd. (e) 65 sq. in.
   (b) 32 sq. in. (d) 136 sq. in.
3. 30 sq. yds.
4. 3750 sq. ft., 11,250 sq. ft.
5. $A = bh$
6. $A = \frac{1}{2} bh$
7. Area should be about 10 sq. in. for both methods.
8. Area should be about 9 sq. in. for both methods.
9.



10. Number of units in $\overline{VS}$ is $(b + x)$

Number of units in $\overline{QV}$ is h

Number of units in $\overline{TS}$ is b

Number of units in $\overline{VT}$ is x

Number of square units in interior of rectangle is $(b + x) h$ which is the same as $bh + xh$ by distributive property. Number of square units in interior of RUS is $\frac{1}{2}$ xh. Similarly for VQT.

Thus, the Number A of square units in the interior of Q R S T is:

$A = (bh + xh) - xh$ or

$A = bh$ by definition of subtraction. (Note: area of right triangle formula is permissible because it depends on the area of a rectangle, which was already known.)

9-9. Right Prisms

Concepts to be developed:

1. Definition of a right prism as a figure obtained from two suitably related polygons of same size and shape lying in parallel planes.
2. Meanings of terms edge, face, vertex, base, height, as applied to a right prism.
3. Method of obtaining volume of the interior of a right prism from the area of the base and the height.

Suggestions.

Section 9-9 has been written from a quite intuitive point of view. In particular, no explicit discussion has been given of the concepts of lines perpendicualr to planes or of perpendicular planes, although both are strictly involved in the idea of a right prism. One point should be brought out clearly by the teacher in classroom discussion. In the case of a right prism, if stood on one base, the upper base is "directly above" the lower base. In fact, if we imagine the prism in this position sliced horizontally (that is, parallel to the planes of the bases) then each such cut is directly above the lower base. Thus, the interior of the prism can be thought of as a series of layers or slabs piled vertically on each other. The fact that such a prism, modeled in wood, can be decomposed by simply cutting it into slabs of wood of the same size and shape and piling them on each other, is tacitly used in the discussion of the volume of the interior of a right prism. For there it is taken for granted that if we construct a slab of volume made of cubes (or parts of cubes) just covering the base, then the top of this slab also just fits inside the prism. It was felt that this fact could better be emphasized by the teacher in class discussion than to try to develop it in detail in the student's manual.

Answers to Exercises 9-9

1. (a) 64 cubic feet (b) 18 cubic cm. (c) 30 cubic in.
2. (a) 30 cubic feet (b) 640 cubic meters (c) 126 cubic in.
3. 135 square feet for each column.
4. (a) $85\frac{2}{3}$ square feet.
   (b) $57\frac{1}{3}$ square feet.
   (c) $37\frac{1}{3}$ cubic feet.
5. $v = Bh$
6. 21 square inches. Shape of base unknown.
7. 180 cubic inches

8. $1\frac{1}{2}$ cubic feet.

10.

Type of Prism	Edges	Faces	Vertices
Triangular	9	5	6
Pentagonal	15	7	10
Hexagonal	18	8	12
Octagonal	24	10	16

Sample Test Questions

True - False

F 1. Nothing has been added to geometry since it was first invented.

F 2. If one angle of an isosceles triangle has a measurement of 66°, one of the other two angles must have a measurement of 66°.

T 3. A statement may be true while its converse is false.

T 4. Pairs of corresponding angles with equal measures are formed when a transversal intersects two parallel lines.

T 5. A statement and its converse may both be true.

F 6. The intersection set of three lines in a plane must be three points.

T 7. Euclid wrote one of the first textbooks in geometry.

T 8. If a triangle has two sides with equal measures, then it has two angles with equal measures.

F 9. All isosceles triangles have the same shape regardless of size.

T 10. The sum of the degree measures of the three angles of a triangle is equal to 180.

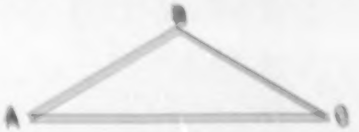
F 11. An equilateral triangle is also a scalene triangle.

F 12. The converse of a false statement is always false.

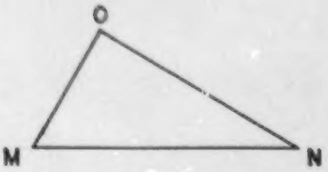
F 13. If a triangle has only two sides with equal measures, it can have three angles with equal measures.

T 14. An equilateral triangle is also an isosceles triangle.

T 15. In the figure at the right, A, B, and C are symbols for the vertices of the triangle.

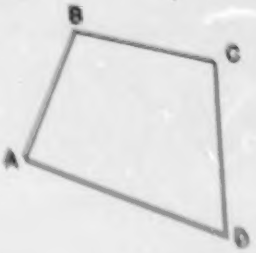


F 16. In the figure shown at the right, if all measures are taken to be equal except those of angle MON and side $\overline{MN}$, as angle MON increases in size the length of $\overline{MN}$ will decrease.



T 17. All parallelograms are quadrilaterals.

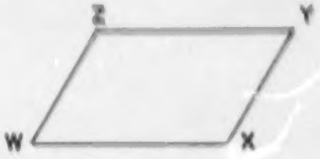
F 18. In the four sided figure at the right, if $m(\overline{AB}) = m(\overline{CD})$ and if AD and BC are parallel, then the figure is a parallelogram.



T 19. A triangle may never include two angles whose measures are 90.

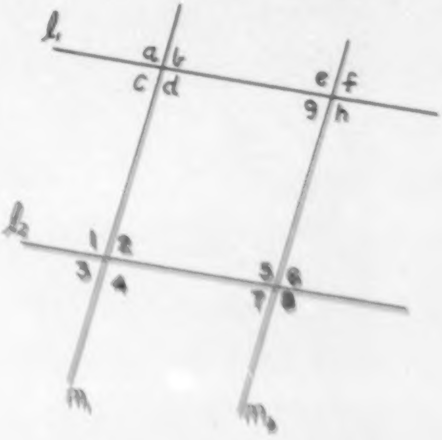
F 20. If the measures of the four sides of a parallelogram are equal, then the figure is always a square.

F 21. In the figure at the right, if WX and YZ are parallel, then WXYZ is a parallelogram.



F 22. It is possible to draw a triangle whose sides measure 4 inches, 2 inches, and 1 inch.

The figure shown at the right consists of two pairs of parallel lines. Use the figure in marking true or false.



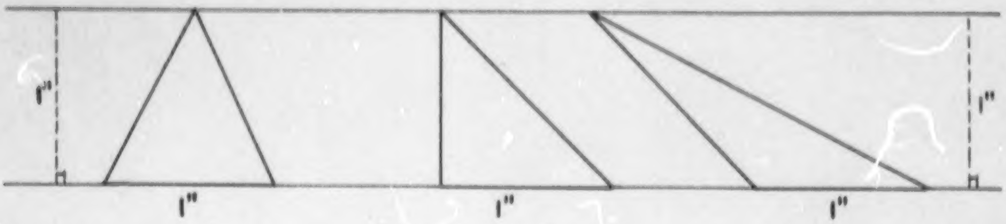
T 23. $m(\angle a) = m(\angle 8)$

F 24. $m(\angle 3) = m(\angle h)$

T 26. The figure contains more than 16 pairs of equal angles.

T 27. If m(∠ 7) = m(∠ 8) then all the measures of the angles shown in the figure are equal.

T 28. The triangles shown below all have the same area.



Multiple Choice

a 1. If a transversal intersects two lines in the same plane and the measures of the corresponding angles are equal, then the two lines are...

(a) parallel lines.

(b) skew lines.

(c) perpendicular lines.

(d) intersecting lines.

(e) none of the above answers is correct.

c 2. If the measure of one angle of a scalene triangle is 50, which of the following statements is always true?

(a) One of the other angles has a measure of 90.

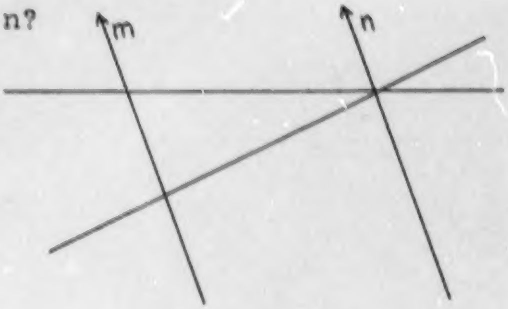
(b) One of the other angles has a measure of 50.

(c) The sum of the measures of the other two angles is 130.

(d) Two of the sides are equal.

(e) One of the other angles has a measure of 130.

b 3. In the figure shown at the right, how many transversals intersect lines m and n?



(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

d 4. If the measure of one angle of a triangle is equal to the measure of the another angle in the triangle...

(a) the measures of the three sides of the triangle are equal.

(b) none of the measures of the sides are equal.

(c) the measures of the sides opposite the angles whose measures are equal are not equal.

(d) the measures of two sides are equal.

(e) none of the above statements is correct.

e 5. If two sides of a triangle have lengths of three inches and four inches, the third side could measure...

(a) one inch.

(b) seven inches.

(c) less than one inch.

(d) more than seven inches.

(e) none of the above answers is correct.

c 6. If three lines are drawn or the same plane and no two of these lines are parallel, the figure formed could include...

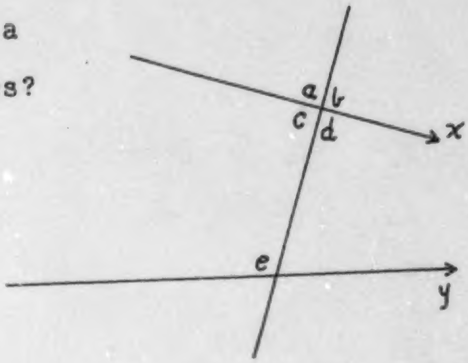
(a) exactly three angles.

(b) exactly two points of intersection.

(c) a triangle.
(d) two closed curves.
(e) a rectangle.

c 7. A geometric plane is...
(a) an airplane.
(b) a carpenter's tool.
(c) a flat surface like that of a window pane.
(d) a curved surface like that of a round lamp shade.
(e) an object like a sheet of plywood.

a 8. In the figure at the right, which angle forms with e a pair of corresponding angles?
(a) a
(b) b
(c) c
(d) d
(e) e



d 9. In the figure above, lines x and y are parallel if...
(a) angle a has the same measure as angle d.
(b) angle c has the same measure as angle e.
(c) angle b has the same measure as angle d.
(d) angle e has the same measure as angle a.
(e) angle a has the same measure as angle b.

e 10. In the figure above, if the measure of angle e is 100, the measure of angle c is...
(a) 100
(b) 80

(c) 20

(d) 10

(e) unknown

a 11. In the figure above, an angle adjacent to angle a is...

(a) b

(b) d

(c) e

(d) angles b , c , and d are all angles adjacent to angle a.

(e) none of the above answers is correct.

e 12. A polygon may have...

I. one side.

II. two or three sides.

III. three, four, or five sides.

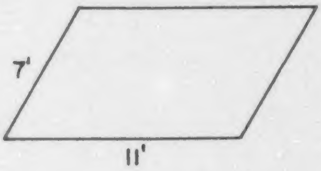
IV. one, three, or five sides.

(a) I and II are correct. (c) III and IV are correct.

(b) II and III are correct. (d) IV is correct.

(e) none of the answers given above is correct.

e 13. The area of the parallelogram shown at the right may be found by...



(a) adding 11 and 7.

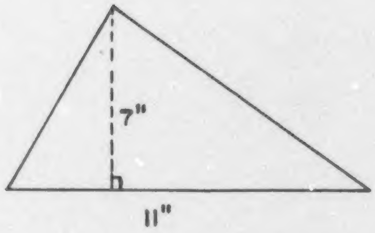
(b) multiplying 11 and 7.

(c) multiplying 11 and 7 and dividing the product by two.

(d) multiplying (11 + 7) by $\frac{1}{2}$.

(e) none of the above answers is correct.

e 14. The area of the triangle shown at the right may be found by...



I. $\frac{11 \times 7}{2}$

II. $\frac{1}{2} \times 11 \times 7$

III. $\frac{1}{2}(11 \times 7)$

IV. $7 \times \frac{1}{2} \times 11$

(a) only I and III are correct.

(b) only II and IV are correct.

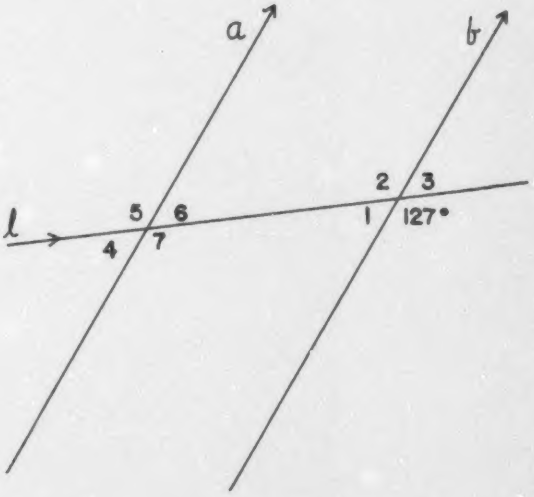
(c) only I and II are correct.

(d) only I, II, and III are correct.

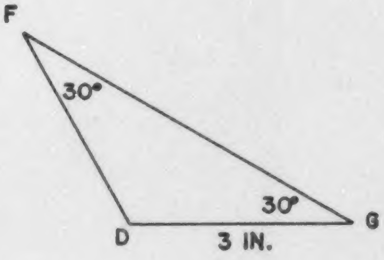
(e) all of the above are correct.

Completion

In the figure at the right lines a and b are parallel and intersect the transversal *l*. What are the measurements of each of the following angles? Note: the measurement of one of the angles is given in the figure.



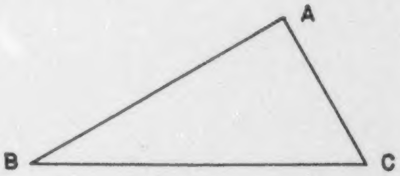
1. angle 6 (53°)
2. angle 2 (127°)
3. angle 1 (53°)
4. angle 5 (127°)
5. In the figure at the right the length of segment $\overline{DF}$ is (3 in.).



6. One of a pair of vertical angles measures 40; the other one would measure (40) .

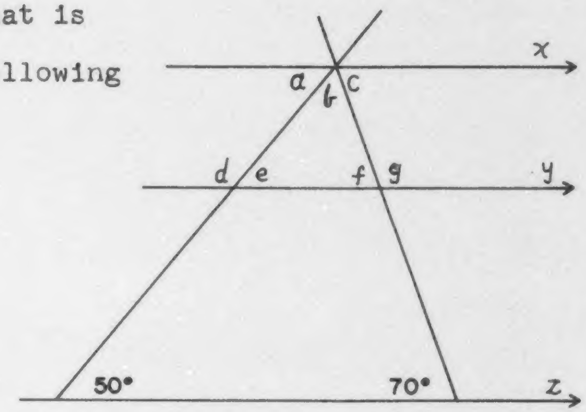
7. Two lines intersect at point A. If the measure of one angle formed is 70, an adjacent angle has a measure of (110) .

8. In the triangle at the right, m(∠ ABC) = 30, m(∠ BCA) = 70). What is the measure of angle CAB? (80)



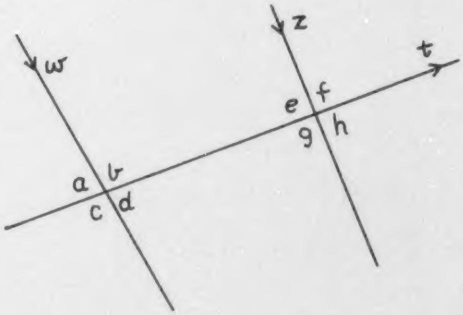
9. Corresponding angles have interiors on (the same) side of the transversal.

In the figure at the right, lines x, y, and z are parallel. What is the measure of each of the following angles?



10. angle a (50)
11. angle b (60)
12. angle c (70)
13. angle d (130)
14. angle e (50)
15. When a line intersects two other lines in distinct points, it is called a (an) (transversal) of those lines.

Using the figure at the right, predict whether lines w and z will be parallel or will intersect. If they intersect, indicate which side of t (above t or below t) the intersection will

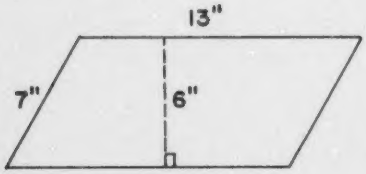


occur. Fill in the space which correctly completes the following statements. (Do not write on this paper. Indicate answers on answer sheet.)

	intersect above t	intersect below t	be parallel
17. If m(∠ a) = 75 and m(∠ e) = 75, then the lines will	( )	( )	(x)
18. If m(∠ b) = 100 and m(∠ e) = 80, then the lines will	( )	( )	(x)
19. If m(∠ c) = 120 and m(∠ g) = 100 then the lines will	( )	(x)	( )
20. If m(∠ d) = 60 and m(∠ e) = 80, then the lines will	( )	(x)	( )

Find the areas of the following:

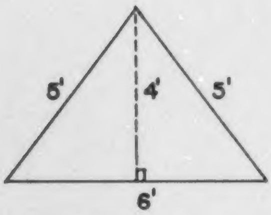
21.



PARALLELOGRAM

(78 sq. in.)

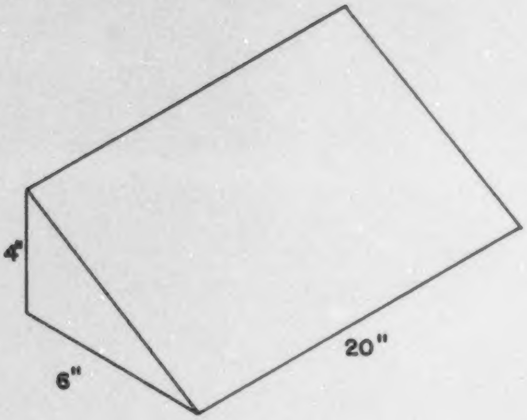
22.



(12 sq. ft.)

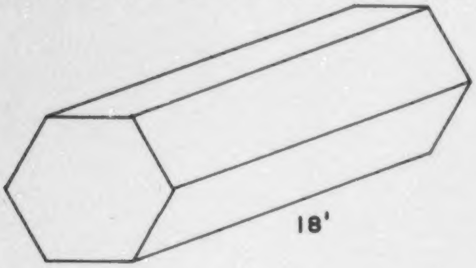
360

23.



(240 cu. in.)

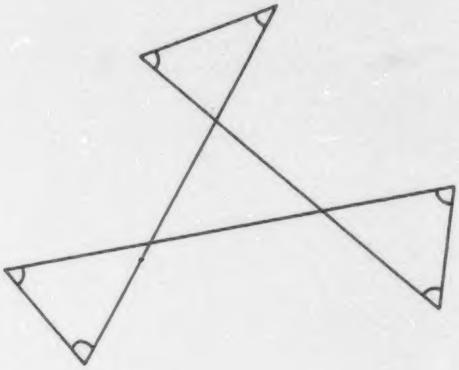
24. Area of the interior of the hexagon is $2\frac{1}{2}$ square feet.



(45 cu. ft.)

Some harder problems

c 25. The plane figure on the right is composed of intersecting straight lines. What is the sum of the measures of the marked angles?



(a) 180

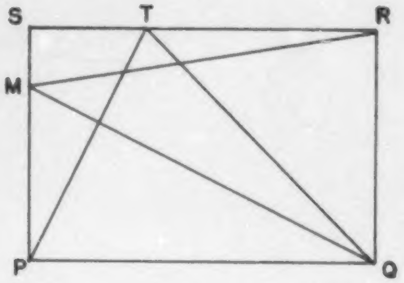
(b) 270

(c) 360

(d) 540

(e) It cannot be determined.

c 26. For the figure on the right PQR is a rectangle. TPQ and MQR are triangles. Let n represent the measure of the area of TPQ and y that of MRQ. If ST = $\frac{1}{3}$SR and SM = $\frac{1}{4}$SP, then $\frac{n}{y}$ = (?) (ST means the measure of the length of segment $\overline{ST}$.)



(a) $\frac{3}{4}$

(b) $\frac{4}{3}$

(c) 1

(d) 3

(e) 4

c 27. If the measure of the vertex angle of an isosceles triangle varies from 40 to 70, then the measure of one base angle varies from

(a) 55 to 70

(b) 70 to 40

(c) 70 to 55

(d) 70 to 110

(e) 140 to 110

CHAPTER 10

CIRCLES

The main purposes of this chapter are

1. to acquaint the pupils with the circle, its length and area, and with some of its elementary applications to cylindrical solids,
2. to develop precision of expression and thought, and
3. to develop geometric awareness and intuition including understanding of appropriate methods of study.

Except for the study of the circle itself, the general purposes of this chapter are like those of chapters 4 and 9. The teacher may choose to reread the introduction to the commentaries for these chapters.

This chapter is written with the intention that it be read and studied by the students.

An effort has been made to use accurate statements concerning the distinction between length and measure of a length. This is done in the spirit of Chapter 7. The word "radius" has two meanings as explained in the text. The number r is not the radius but rather the measure of the radius.

Because in ordinary speech distinctions have not been made between length and measure of a length, many teachers may find themselves interchanging these terms. An important distinction is that we multiply numbers but not lengths as such. For many purposes this idea is not too important for seventh grade pupils. Teachers are advised to note such subleties of thought and to make some effort to use precise language. They should not be alarmed at their own failure to do so consistently. They should not insist on more than reasonable precision from students.

Time schedule. A time estimate is as follows:

First week	Sections 10-1 through 10-4
Second week	Sections 10-5 through 10-7
3 days of third week	Sections 10-8, review problems, and text evaluation

The sections most likely to take two days each are 10-4, 5, and 6. The time schedule given is designed for an average class. Slower classes may take a bit longer. Accelerated classes might do the chapter in two weeks, about a section a day.

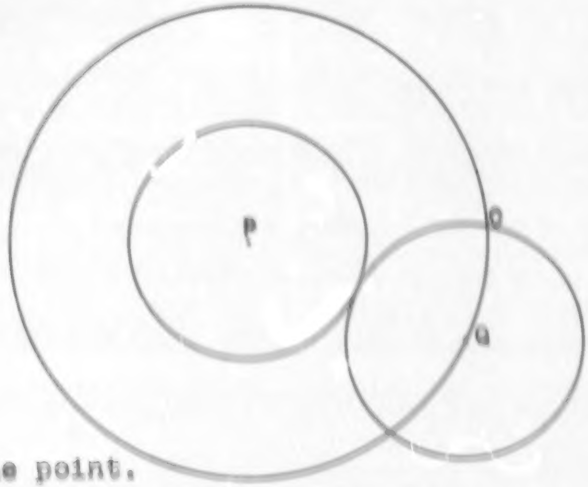
10-1. Introduction

You may wish to begin this chapter by calling attention to the prevalence of the circle in industrial and in decorative design. Some pupils enjoy bringing in pictures illustrating the use of the circle. In the history of civilization, discovery of the use of the wheel is ranked in importance with discovery of a method for producing fire.

Exercises 10-1 are planned to provide practice in following directions for drawing and labeling figures, and in use of the compass. It is important that each pupil have a compass and learn to use it correctly and with some dexterity. In following the directions for drawing figures, pupils should learn to label the figure at each stage, as subsequent directions are given in terms of the labels specified.

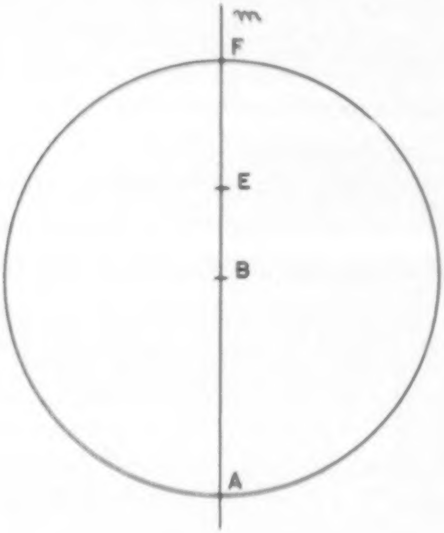
Answers to Exercises 10-1

1. (a) and (b)



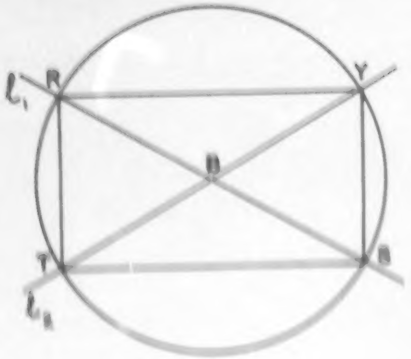
   (c) Intersection is one point.

2. (a) - (e)



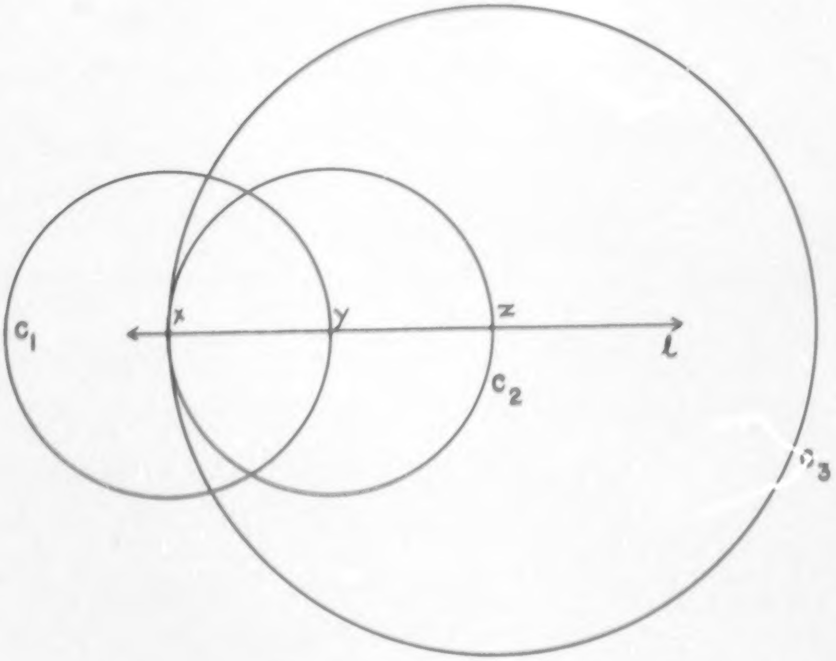
   (f) Point A and Point F.

3. (a) - (c)



(d) RTSY is a rectangle. (If the diagonals of a quadrilateral are the same length, and also bisect each other, then the quadrilateral is a rectangle.)

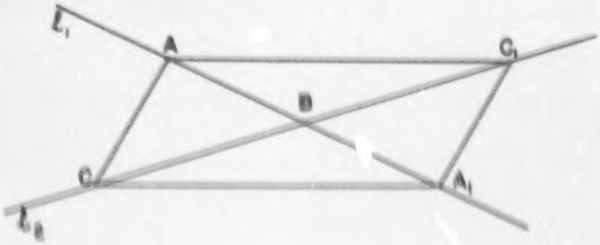
4.



(e) Intersection set is two points.

(f) Point X

*5.



(e) ACA_1C_1 is a parallelogram. (If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.)

(f). Eight triangles: ACB, CBA_1, A_1BC_1, C_1BA, ACA_1, A_1C_1A, C_1AC, CA_1C_1 .

10-2. Interiors and Intersections

Each pupil should be repeatedly encouraged to adopt the study pattern elaborated in the third and fourth paragraphs. It is vital that he translate the written idea into a diagram in a step-by-step manner, while he is reading. This cannot be overemphasized, for it applies throughout geometry.

The single point Q lies on the circle and also on $\overrightarrow{PQ}$. Two points of the circle lie on $\overrightarrow{QP}$, of which one is Q and the other needs a label. Any name is suitable; perhaps R will be suggested naturally. Intuitively R is directly opposite Q on the circle; a formal description of R is the point, other than Q, in the intersection of the circle and the line $\overleftrightarrow{PQ}$.

"Radius" fits the blank which calls for an answer. The intersection of D and $\overrightarrow{PQ}$ is the segment $\overline{QR}$ (if "R" was the label used above), while the intersection of $\overrightarrow{PQ}$ and the circle consists of only two points, Q and R.

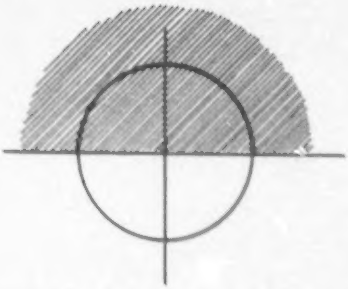
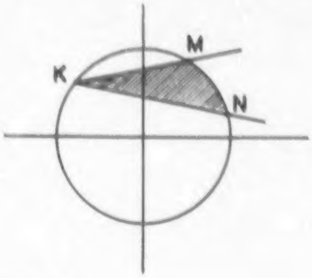
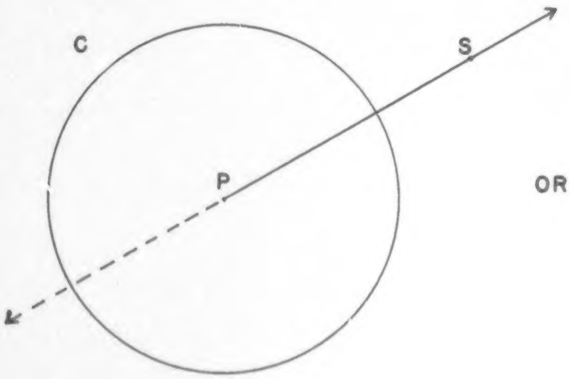


Figure 10-2, modified to show H and its intersection with the circle, is pictured. None of the points A, G, P belongs to the intersection, but F does. We cannot count all the points of the intersection. One possible choice for M, N, K is shown. The intersection of the circle and the angle consists of only the three points M, N, K. The intersection of the two interiors has been shaded in the figure.

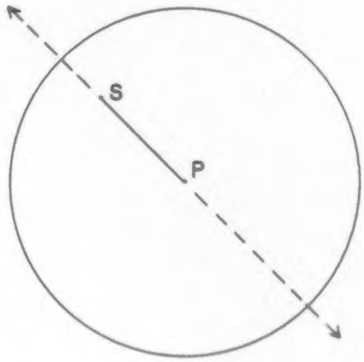


Answers to Exercises 10-2

1. (a) 1.



OR



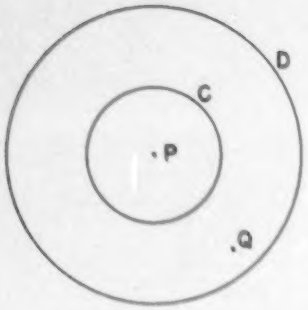
(b) 2.

(c) No.

(d) If S lies in the interior of the circle, the number of points is 0; if S lies on the circle or outside the circle, the number is 1.

2. Yes.

3.



4. (a) Same as interior of angle BPF.

(b) Intuitively, the quarter-circle in the upper right, excluding the endpoints B and F.

(c) Four of the quarters may be identified. Besides the one in (b), we have the intersection of the circle and the interior of angle APF, in the upper left; the intersection of the circle and the interior of $\angle APG$; the intersection of the circle and the interior of $\angle BPG$. In each case, a quarter can also be described as the intersection of the circle and two halfplanes; for example, the lower left quarter is the intersection of the simple closed curve AFBG, the A-side of $\overleftrightarrow{FG}$, and the G-side of $\overleftrightarrow{AB}$.

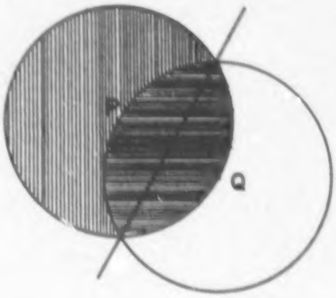
(d) Four portions of the circle might be called halves. The upper half, for example, is the intersection of the circle and the halfplane H. In Section 4, we learn that this intersection, together with its endpoints A and B, is a semicircle. Another is the lower half, the intersection of the circle and the G-side of $\overleftrightarrow{AB}$. Another is the intersection of the circle and J. The fourth one shown is the intersection of the circle and the A-side of $\overleftrightarrow{FG}$. In each of the four cases, the curve together with its endpoints is a semicircle. The pupils may wish to include the endpoints when they describe a "half" of the circle. This problem helps blaze the trail for certain notions in Section 4.

5. (a) Consists of two points.

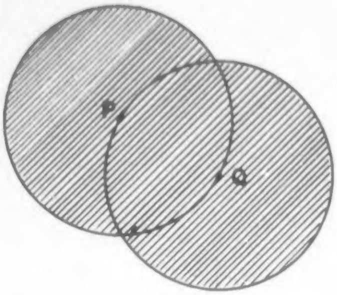
(b) Yes; no; two points determine just one line.

(c) Horizontally shaded in the figure.

(d) Vertically shaded in the figure.



(e)

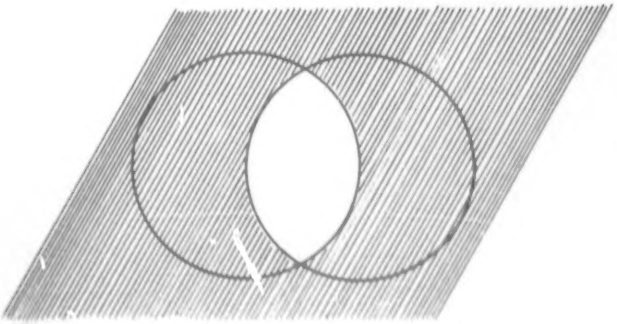


6. (a) Empty set.

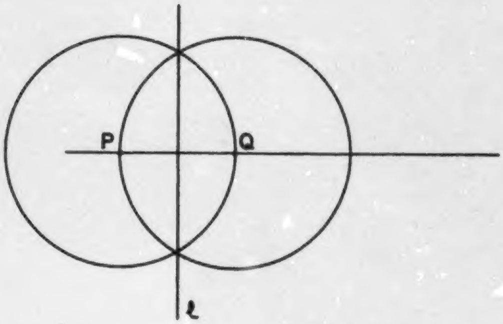
(b) Intersection of the interior of the outer circle and the exterior of the inner circle.

*7. Same as exterior of the outer circle.

*8.



*9. The lines ℓ and $\overleftrightarrow{PQ}$ are perpendicular.



10-3. Diameters and Tangents

The six radii shown in the figure are $\overline{PW}$, $\overline{PA}$, $\overline{PM}$, $\overline{PV}$, $\overline{PB}$, and $\overline{PN}$.

Since $d = 2r$, $r = \frac{1}{2}d$.

A circle does not need to be "on top" of a line in order for the line to be tangent to the circle.

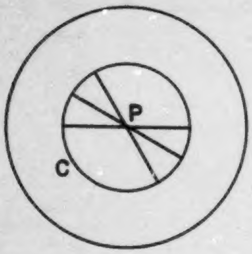
Answers to Class Questions.

1. The first mentioned intersection is the empty set, the second is the interior of the circle.
2. Yes.
3. No.
4. Four; one; E, F, S.
5. One; one.

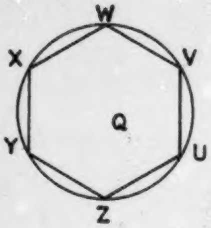
Answers to Exercises 10-3

1. Thirty-two: each of the 8 circles has 4 lines tangent to it.
3. 21.

4.



5.



(g) Should be same (but the drawing is hard to do with enough care).

(h) A diameter of the circle.

6. (a) Consists of four points R, S, T, U.

(b) Consists of Point T.

(c) Point of tangency.

(d) 3: HE, EF, FG.

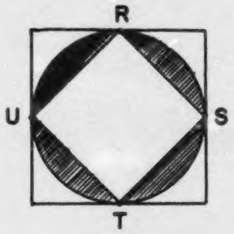
(e) R, S, T, U.

7. (a) O.

(b) Empty set.

(c) Intuitively it is composed of the four corner regions. An attempt on the pupil's part to give a careful description should convince him that the very best description is the one quoted in the problem: "the intersection of the exterior of the circle and the interior of the square EFGH." A purpose of this problem is to point out the advantages of our vocabulary in enabling us to say just what we mean with the greatest ease.

(d) As in the preceding part, the best word description is given in the problem itself. The intersection is shaded in the figure.



*8. (a) Yes. The pupil's answer may be based on the notion of symmetry, but there are many ways in which he can express his thoughts.

(b) Yes.

(c) Right angle.

(d) Perpendicular.

*9. Lines are perpendicular.

*10. Each diameter is the union of two different radii on one line. If the intersection of two segments on one line consists of a common endpoint, then the length of their union is the sum of the lengths of the two segments. Since all radii have the same length, the measure of a diameter is twice the measure of a radius, that is, twice the measure of the radius. Since this applies to all diameters, all of them have the same length.

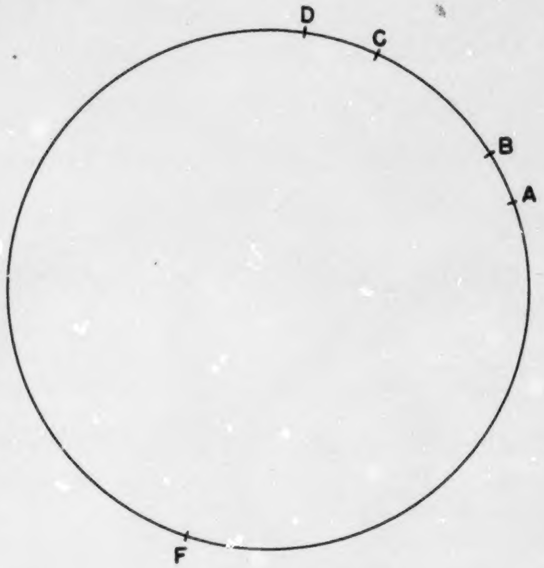
10-4. Arcs and Central Angles

Definitions and qualifying descriptions which are especially important in this section are: arc, symbolism for arcs, central angle, arc-degree.

"Straight angle" may be a troublesome notion. A "straight angle" or a "central angle of 180°" does not fit our description of "angle" because we have defined angle as a set of points that consists of two rays which have the same endpoint but do not lie in the same line. This precise definition accounts for the fact that in the text the term "special" is applied to the mention of the "straight angle."

Answers to Exercises 10-4

1. (a) none, (b) none, (c) A, B, (d) none
2. (a) m(ABC) = 118 (d) m(BCD) = 121
   (b) m(ABCD) = 136 (e) m(CDE) = 101
   (c) m(DE) = 83
3. (a) - (d)



   (e) 35°

4. Sketch a circle and sketch a diameter. Drawing in a few lines through the center which intersect both semi-circles should show that a one-to-one correspondence can be established between the set of points on each of the two semi-circles. This method is not unique. Another method involves perpendiculars drawn to the diameter. These set up another one-to-one correspondence between

the sets of points of the two semi-circles. Originality in ways of looking at things should be encouraged.

5. Quarter of a circle: 90°
   One-eighth of a circle: 45°
   One-sixth of a circle: 60°
   Three-fourths of a circle: 270°

6. (a) Yes
   (b) 6
   (c) 60

7. (a) $\overset{\frown}{BC}$
   (b) $\overset{\frown}{DF}$
   (c) $\overset{\frown}{CD}$
   (d) D (the Point D)
   (e) $\overset{\frown}{DE}$

10-5. Length of a Circle

The relation between the length of a circle and the length of its diameter is developed in two ways. The first method consists of measuring the circumferences and diameters of circles, examining the measures for a possible relation, and observing that, in any circle, the ratio $\frac{c}{d}$ is approximately the same number. It is good procedure to have the pupils report their measurements, tabulate them, and then propose relations which might exist. This is to be preferred to guiding them too soon and too firmly toward the ratio.

The second method is important as a basis for underlining the notion that π is a number, and that the symbol "π" is to be regarded as a numeral. This is the first number of the set of irrational numbers the pupils have met, and they should develop the notions that (1) such numbers correspond to points on the number line, and (2) that a decimal expression for the number, to any desired precision, may be written. Although π is not a rational number, the assumption is tacitly made here that it behaves like an ordinary number in combining with other numbers; that is, the commutative, associative, and distributive properties for addition and multiplication hold. Many pupils are interested in the fact that mathematicians have studied the properties of this number for centuries, and continue to do so; and are entertained by seeing the decimal for π to so many decimal places.

The process of describing the position of π on the number line by defining the segment on which it lies is important as an illustration of a method of analysis, and also for the illumination it sheds on the significance of successive places in decimal notation. It is recommended that this material be worked through very carefully.

In the set of Exercises 10-5-b, your attention is directed to Problem 10. The idea of using π as a numeral, and expressing answers in terms of π, is a useful and important one, and we recommend that all pupils be assigned this problem. In general, it is desirable that all answers be expresses first in terms of π, with the decimal evaluation the last step in computation when a decimal answer is required.

The authority for the statement that the decimal expression for π has been computed to 10,000 places in an article by F. Genuys in Chiffres 1 (1958), 17-22.

Pupils may be amused by a mnemonic device for remembering the first figures in the decimal for π. It is a rhyme in which the number of letters in each word indicates the digit:

"See, I have a rhyme assisting
My feeble brain, its task sometime resisting."

Answers to Exercises 10-5-a

1. (a) - (c) Answers will depend on measurements obtained.

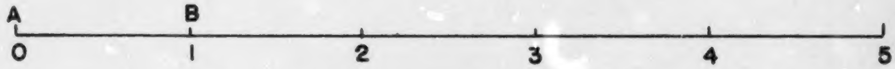
(d) Differences are not the same.

(e) Ratios will vary, but should be distributed around 3.1 or 3.2. If a correctly computed ratio is larger than 3.4 or smaller than 2.9, suggest that the object be measured again.

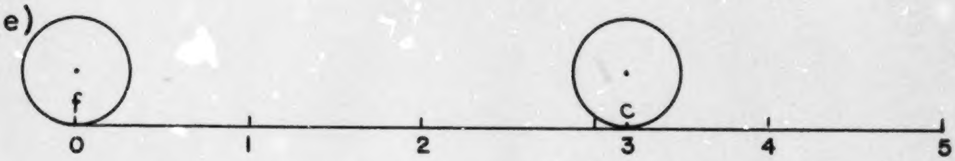
(f) 3.1.

(g) Should be nearly the same.

2. (b) - (c)



(d) - (e)



(f) Between 3 and 4. Estimates will probably vary from 3.0 to 3.3.

(g) $\overline{AD}$

(h) $\overline{AB}$

(i) Length of the circle is about 3.0 - 3.3 times the length of the diameter.

(j) Results should agree fairly closely.

Answers to Questions in paragraph, The Number π

Statement should be:

$+ \ 3.141 + .000592 \ldots$ Therefore, $\pi > 3.141$ and $\pi < 3.142$.

$(3.141 < \pi < 3.142)$.

Endpoints of segment IV should 3.14 and 3.15. Subdivisions should be labeled .001, .002, etc. π is on the segment whose endpoints are 3.141 and 3.142 (labeled .001 and .002).

Answers to Exercises 10-5-b

Computed results are stated first, then the answer justified by the precision of the measurements given.

1. (a) Diameter, ≈ 8.4 cm. Circumference, ≈ 26.376 cm., or 26 cm.
   (b) Radius, ≈ 2.8 in. Circumference, ≈ 17.584 in., or 18 in.
   (c) Radius, ≈ 3.98 ft. or 4.0 ft. Diameter, ≈ 7.96 ft., or 8.0 ft.
2. (a) .025, .001, .017
   (b) $\frac{22}{7}$
3. 37.68 in., or 38 in.
4. 19.7 in., or 20 in. Yes.
5. 94.2 ft., or 94 ft.
6. 3.82 ft., or 3.8 ft.
7. (a) 62.0 in.
   (b) 7.75 in., or 8 in.
   (c) 45 degrees.
   (d) .17 in., or .2 in.
8. (a) Length of Circle C ≈ 43.4 in., or 43 in.
   Length of Circle D ≈ 31.0 in., or 31 in.
   (b) $m(ST) \approx 70$
   $m(QR) \approx 70$

(c) $\frac{7}{36}$ $\frac{7}{36}$

(d) 8.4 in., or 8 in.

6.0 in., or 6 in.

9. (a) 6.2832

(b) 9.42477

10. (a) Diameter, 54 in. Radius, 27 in.

(b) $(13 \cdot \pi)$ in.

(c) $(7.2 \cdot \pi)$ in.

11. Circumference of Circle C is three times circumference of Circle D.

12. (a) If the circumference is increased by 1, the new diameter is $\frac{c + 1}{\pi}$, or $\frac{c}{\pi} + \frac{1}{\pi}$. The radius has thus been increased by $\frac{1}{2}\left(\frac{1}{\pi}\right)$ ft. which amounts to about 1.9 inches. This would permit a mouse to crawl under the band

(b) No. If the band is made $6\frac{1}{4}$ feet longer than the original tightly fitting band, the new radius of the enlarged band will be about 1 foot greater than the radius of the original band, allowing 1 foot of crawl space.

(c) $\frac{c}{2\pi} = r$; distributive property.

$$d = \frac{1}{\pi}(C + 1) = \frac{c}{\pi} + \frac{1}{\pi}.$$

10-6. Area of a Circle

Chapter 7 explains the basic method for *finding* the area of a region. In the case of a circle we prefer to have a method for *computing* the area in terms of the radius. Three approaches to the development of the relationship $A = \pi r^2$ are presented in this section. The first is an application of the basic method of measurement and yields a rough approximation. The second offers some plausibility of the presence of the coefficient π in the relationship $A = \pi r^2$. The third appears in Problem 9 at the end of the section and, by virtue of its activity, may appeal to the children.

The questions in the fourth, fifth, and sixth paragraphs may be appropriate for class discussion. The measure of $\overline{EB}$ is $2r$. The measure of $\overline{AB}$ is $2r$, since ABEF is a square and all its sides have the same length. The measure of the interior of ABEF is $4r^2$. This is 4 times the measure of the interior of AZPV. The interior of the circle has less area than the interior of the square and, therefore, should have an area which is less than $4 \cdot r^2$ units. However, inspection should reveal that the interior of the circle is more than two-thirds as large as the interior of the square; as for three-fourths, this is a rather subjective decision and different children may answer differently. The exact ratio $\frac{\pi}{4}$ is only slightly larger than $\frac{3}{4}$.

In counting the units of area, we may take advantage of the symmetry and count in just one quarter. In the upper right quarter, for example, 69 square regions lie entirely within the circle (by rows, from the top to the middle of the figure, 0,4,6,7,8,8,9,9,9,9). An additional 10 square regions have the property that the larger portion is inside the circle (again, by rows, from the top, 3,1,1,1,0,1,0,1,1,1). It is not difficult to find an approximate balance between the portions of these outside the circle and the small pieces of the remaining 7 regions which contribute to the inside. (The 7 regions, by rows, from the top, occur: 2,1,1,0,1,1,1,0,0,0.) Note that in each quarter of the figure there are drawn two square regions of which no part lies inside the circle. After counting (69 and 10), we assign the measure 79 to the area of a quarter. Thus, for the whole circle, the measure of its area is 316. Note that the value 79 is the best possible counting number that can be expected, since the theoretical measure is 78.54, to the nearest hundredth.

When the children draw their own circles and measure with this counting procedure, there may be considerable variation in the results because of inadequate drawing performance and/or equipment. Most of the development should be based on the professionally drawn diagram in the book.

There is an excellent way to test the child's understanding of the second treatment of the relationship $A = \pi r^2$. Can he handle the discussion when the number 60 used in the text is modified to some other choice--say, 30 or 180 or 100? After all, the number 60 is not significant as far as the end result is concerned. Any one of many other numbers could have been selected.

Answers to Exercises 10-6

1.

r	d	c	A
20	40	124	1240
4	8	24.8	49.6
0.015	0.030	0.093	0.0006975
111	222	688.2	38195.1

2. Since $49 < 50.24$, the circular skillet.
3. 36π square-inches, or approximately 113 square-inches.
4. First; yes.

5. Approximately 70 quadrillion square-kilometers (70,000,000,000,000,000).

6. $40 \cdot 30 - 7^2 \cdot \pi$; to the nearest square-foot, 1046.

7. (a) 16π.
   (b) 16.
   (c) 4.
   (d) 8.
   (e) 8π.
   (f) 4π.
   (g) $8 + 4\pi$.

8. 3.

9. $22\frac{1}{2}$; parallelogram whose base is πr units, and whose altitude is r units. This activity requires considerable care in the construction and cutting of the diagram. Result should be πr^2.

10. (a) Chain at corner.
    (b) 785.4 square-feet.

Answers to Exercises 10-7

The numerical answers given look more precise than the indicated accuracy of the original measurements would justify. It would be reasonable to round some of them off.

1. $V \approx 3391.2$ or, Volume is 3391.2 cu. ft. (3391, 3390, or 3400 could be used).

2. Volume is 2π cu. ft., 6 cu. ft.

3. 47 gallons (or 45 gallons if one computes from 6 cu. ft.)

4. $V \approx 3768$. Or Volume is about 3770 cu. in.

5. $V \approx 90\pi$.

6. $V \approx 180\pi$.

7. $V \approx 360\pi$.
8. $V \approx 720\pi$.

*9. It doubles. (V is multiplied by two.)

It quadruples (V is multiplied by four.)

V is multiplied by eight.

10. 52.31 cu. in. or about 52 cu. in.

Answers to Exercises 10-8

The numerical answers given for problems 5, 6, and 7 imply much more accuracy than is justified by the numbers used. It would be reasonable to round them off.

1. Rectangle (or rectangular region); altitude or height; base circle.
2. (Construction problem).
3. Use tape to find length of base circle directly.
4. Yes.
5. Lateral surface area is 75.36 sq. centimeters.
6. Total Area is 89.49 sq. centimeters.
7. About 10.4 sq. meters; about 8.3 sq. meters.
8. 3.5 gallons.

Answers to Exercises 10-9

1. a, b, c.

Diagram	(a)	(b)	(c)
(1)	circle	circumference	$c = 2$[illegible]r
(2)	rectangle	area	$A = lw$
(3)	triangle	area	$A = \frac{1}{2}bh$
(4)	circle	area	$A = \pi r^2$

(5)	rectangular prism	volume	$V = lwh$
(6)	circle	circumference	$c = \pi d$
(7)	cylinder	lateral area	$L = 2\pi rh$
(8)	triangular prism	volume	$V = \frac{1}{2}b_t h_t h_p$
(9)	parallelogram (or rectangle)	area	$A = bh$
(10)	cylinder	volume	$V = \pi r^2 h$
(11)	rectangular prism	lateral area	$L = 2\ lk + 2\ hw$
(12)	cylinder	total area	$T = 2\pi rh + 2\pi r^2$

2. (a) Numerals to be circled are 2, $\frac{1}{2}$, and π wherever they occur.

(b) Symbols r, l, w, b, h, d, b_t, h_t, h_p, wherever they occur.

(c) Number of arrows below the ruler is the number of dimensions of the figure.

(d) See answer to Problem 2 (c). Number of measures indicates number of dimensions.

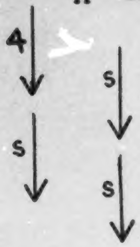
(1) 1 (3) 2 (5) 3 (7) 2 (9) 2 (11) 2
(2) 2 (4) 2 (6) 1 (8) 3 (10) 3 (12) 2

(e) Number of measures in formula which is a product indicates number of dimensions. In (11) and (12), number of measures on one branch indicates number of dimensions.

(f) "B" above bottom arrow in (5), (8), and (10).

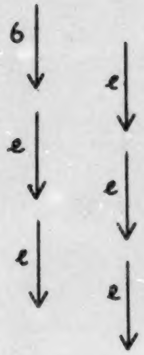
(g) Square: $p = 4s$

$A = s^2$

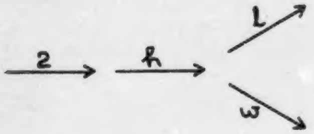


Cube: $S = 6e^2$

$V = e^3$

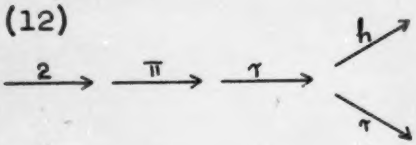


(h) (11)



$L = 2h(\ell + w)$

(12)



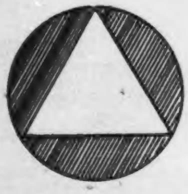
$T = 2\pi r(h + r)$

3. (a) 68° (b) 150° (c) 93° (d) 126° (e) 85°

4. (a) 519 square meters

(b) 1257 square meters

(c)



(d) 738 Square meters

(e) 246 square meters

5. (a) 66 meters; 144 square-meters.

   (b) 119 centimeters; 814 square-centimeters.

   (c) $80 + 20\pi$ feet; $480 + 200\pi$ square-feet.

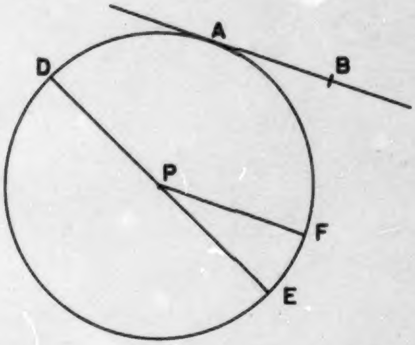
   (d) $62 + \frac{3\pi}{2}$ millimeters; $84 + \frac{9\pi}{4}$ square-millimeters.

6. 72.8 cu. ft.

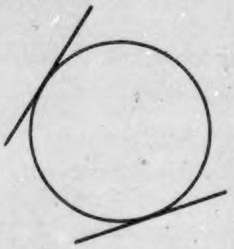
7. 113.6 sq. ft.

Sample Test Questions

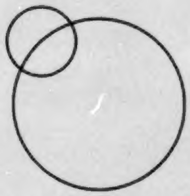
1. In the figure at the right, find an illustration for each term in the column at the left, and write its name on the line after the term.



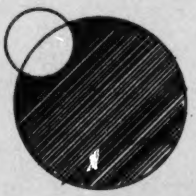
		Answer
(a)	Diameter	$\overline{DE}$
(b)	Radius	$\overline{PF}$ or $\overline{PE}$ or $\overline{PD}$
(c)	Arc	$\overset{\frown}{EF}$ or $\overset{\frown}{AF}$ or $\overset{\frown}{DA}$ or $\overset{\frown}{DF}$, etc.
(d)	Tangent	$\overleftrightarrow{AB}$
(e)	Semicircle	$\overset{\frown}{DAE}$ (or $\overset{\frown}{DE}$)
(f)	Obtuse angle	$\angle DPF$
(g)	Two line segments of equal length	$\overline{DP}$, $\overline{PF}$, $\overline{PE}$
(h)	Central angle	$\angle DPF$, $\angle FPE$
(i)	Two points which separate A and E	D and F
(j)	Acute angle	$\angle FPE$

2. Ellen's belt is 21 inches long and just fits around her waist. If her waist were a perfect circle, what would its diameter be?	Answer: 6.7 in.
3. A value for π used by the ancient Egyptians is $\frac{355}{113}$. How different is this from the value we use, to two decimal places?	Answer: Same to two decimal places.
*4. A merry-go-round has three rows of horses, the outer row being 6 feet farther out than the inner row. If you sit on an outside horse, how much farther do you ride in one turn of the merry-go-round than if you sit on the inside horse?	Answer: 12 π feet or 37.68 feet.
5. The diameter of a certain circle is 60 centimeters. What is the area of this circle? (Use the decimal approximation to π to the nearest hundredth.)	Answer: 2826 square centimeters.
6. Draw a circle and two segments such that each of the segments is tangent to the circle.	One possible answer: 

7. Shade the intersection of the interior of the longer circle and the exterior of the shorter circle.

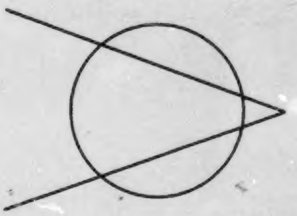


Answer:

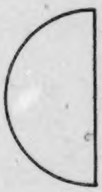


8. Draw a circle and an angle such that their intersection consists of four points.

One possible
Answer:

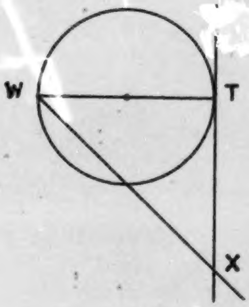


*9. The simple closed curve in the figure consists of a semicircle and a segment. The radius of the circle is r units of length. Find the area of the interior of the simple closed curve. (The answer should be expressed in terms of r.)



Answer:
$\frac{1}{2}\pi r^2$ units of area.

10. The segment $\overline{WT}$ is a diameter of the circle. The lines $\overleftrightarrow{TX}$ and $\overleftrightarrow{WT}$ are perpendicular. The segment TX is 3 feet long. The measurement of the angle TWX is 45 degrees.



(a) What is the measure of $\angle WXT$ in degrees?	Answer: 45.
(b) Find the radius of the circle.	Answer: $\frac{3}{2}$ feet.

Sample Test Questions from 10-4

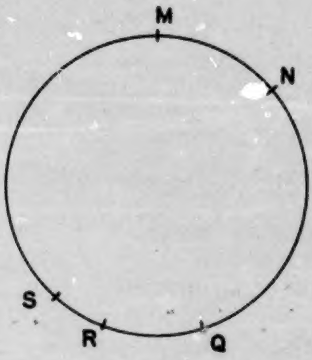
11. Construct a circle with radius approximately 2 inches. Mark off and label arcs with the following measures. Mark off the points in a clockwise path around the circle after starting anywhere on the circle with M.

(a) $m(\widehat{MN}) = 50$

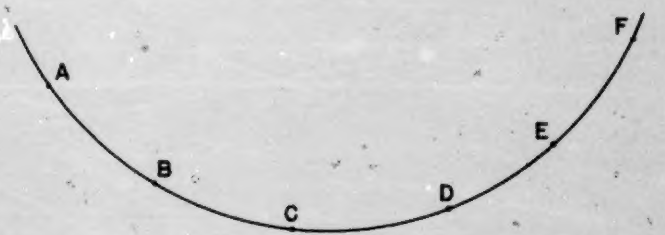
(b) $m(\widehat{NQ}) = 110$

(c) $m(\widehat{NR}) = 150$

(d) $m(\widehat{RS}) = 20$



12.

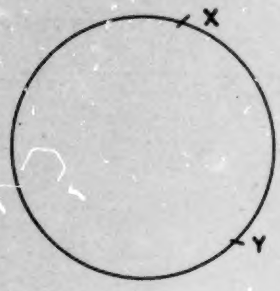


Given the arc shown above, determine the following:

	Answer:
(a) $\widehat{BE} \cap \widehat{CF}$	(a) $\widehat{CDE}$
(b) $\widehat{AF} \cap \widehat{DE}$	(b) $\widehat{DE}$
(c) $\widehat{AC} \cap \widehat{BF}$	(c) $\widehat{BC}$
(d) $\widehat{CE} \cap \widehat{DE}$	(d) $\widehat{DE}$

13. With the aid of a sketch, define "arc".

Answer



The points x and y determine an arc on the circle. The arc includes its end-points x and y. These two end-points determine two arcs.

14. Find the volume of the inside of a pipe whose length is 80" and the radius of whose inside base circle is $\frac{1}{4}$". Use $\pi \approx 3.14$.

Answer:
15.70 cu. in.

15. Find the total surface area of a cylinder the radius of whose base circle is 10 centimeters if the altitude of the cylinder is 100 centimeters. Use $\pi \approx 3.1$.

Answer:
6820 square centimeters.

16. Find the amount of paper needed to make labels (without overlapping them) for 3 cans each of height 6" and of base circle radius 2" . Use $\pi \approx 3.14$.

Answer:
226.08 sq. in.

17. Find the volume of a cylindrical solid the diameter of whose base circle and whose height are each 3 meters. Leave your answer in terms of π .

Answer:
$\frac{27}{4}\pi$ or 6.75π cu. meters.

CHAPTER 11

STATISTICS AND GRAPHS

Objectives

1. To develop some facility in the elementary uses of statistics through a study of some data collected by the government, by business, and by the students themselves.
2. To develop an appreciation of the important use of mathematics in social studies, thereby motivating and strengthening both.
3. To illustrate how data may be tabulated and the information contained in the data represented by different kinds of graphs.
4. To give instruction in reading information from a graph for which no table of data is given.
5. To develop some skill in calculating arithmetic means, medians, and average deviation from the mean.

Suggestions

1. Do not expect all pupils to do all the graphs. For example, in Exercises 11-4 a suggestion is made in this manual for distributing the work among the pupils in the class.
2. The National Industrial Conference Board, 460 Park Avenue, New York 22, will supply good supplementary

material for this chapter at your request.

3. A very informative and pleasantly written book on statistics is Darrell Huff's How to Lie with Statistics, W. W. Norton and Company, Inc., New York, New York.

Many of the data used in the chapter were found in the Statistical Abstract of the United States.

Answers to Exercises 11-1

1. The general trends in the data show that the population has been increasing since 1790. The per cent of increase has been less since 1860 than before that time.
2. From 1800 to 1810. High immigration rate and beginning of Westward Movement.
3. From 1930 to 1940. The depression of the 1930's.
4. Population increase was due in part to large number of immigrants from Ireland. Famine was due to failure of potato crop in Ireland in 1845, 1846, and 1847.
5. 26.0.

Answers to Class Discussion Questions 11-2

1. More between 1900 and 1910 than between 1800 and 1810. This is seen more easily from the graph than from the table of data.
2. No.

3. The piece of the broken line from 1810 to 1820 would be horizontal.

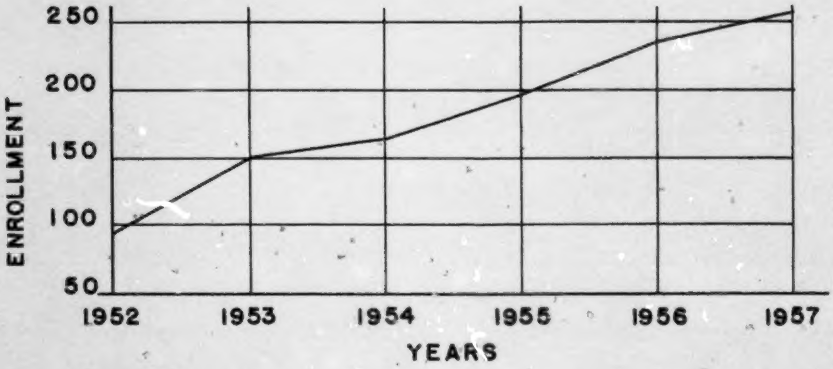
4. 1945: 140 million; 1895: 68 to 70 million. Increase 70 to 72 million.

5. About 170 million.

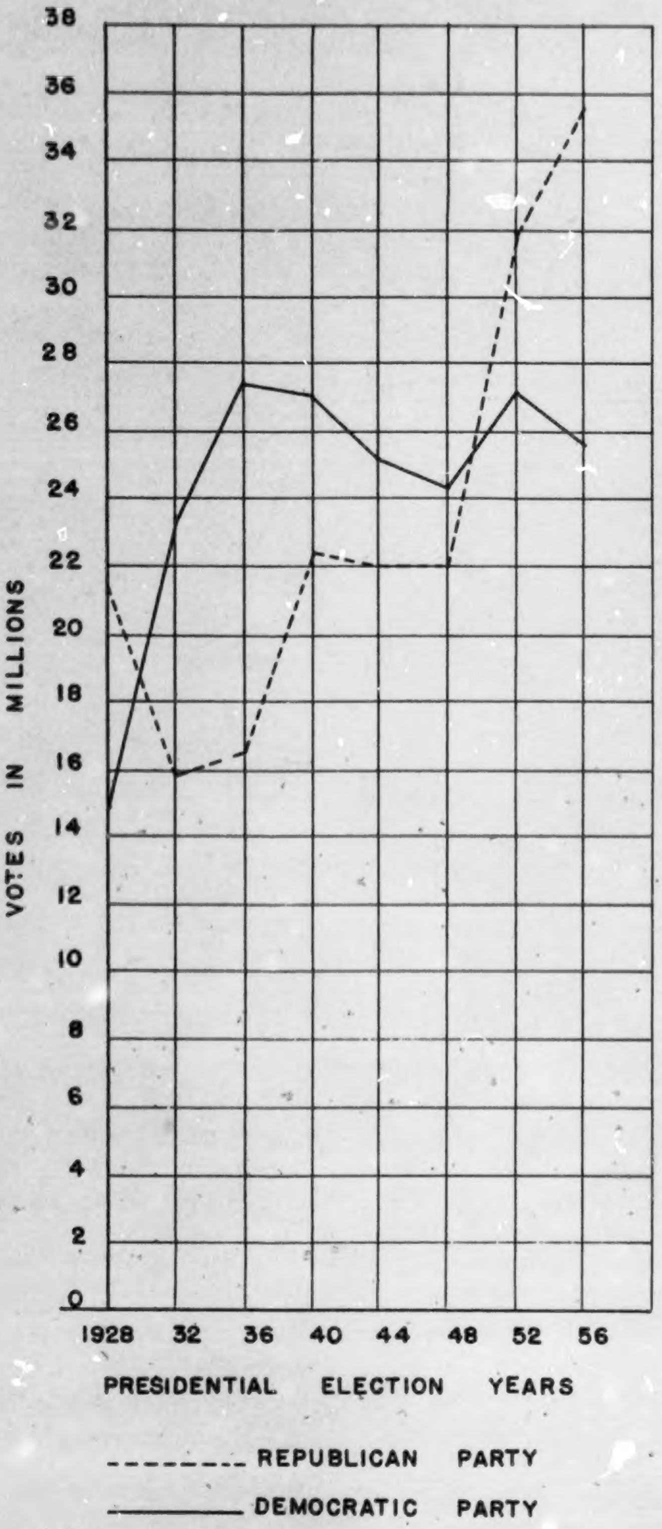
Answers to Exercises 11-2

1.

Enrollment in Franklin Junior High School, 1952-1956



2. (a) In this problem, it is suggested that a few of the pupils work together in making the graph on the chalkboard. Do not have the pupils make the graph on paper. The table called for in 2(b) can be put on the chalkboard by another group of pupils. 2(c) is a good review problem for all pupils. 2(d) and 2(e) can be discussed by the pupils while the graph is on the chalkboard.


VOTES IN MILLIONS
38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
1928
32
36
40
44
48
52
56
PRESIDENTIAL ELECTION YEARS
-------- REPUBLICAN PARTY
———— DEMOCRATIC PARTY

2. (b)

(D) Democrat
(R) Republican

Year	Elected	Unsuccessful Candidate
1928	Hoover (R)	Smith (D)
1932	F. D. Roosevelt (D)	Hoover (R)
1936	F. D. Roosevelt (D)	Landon (R)
1940	F. D. Roosevelt (D)	Willkie (R)
1944	F. D. Roosevelt (D)	Dewey (R)
1948	Truman (D)	Dewey (R)
1952	Eisenhower (R)	Stevenson (D)
1956	Eisenhower (R)	Stevenson (D)

(c) $\frac{22.3 + 26.8}{131.7} = 37.3\%$

Answers to Exercises 11-3

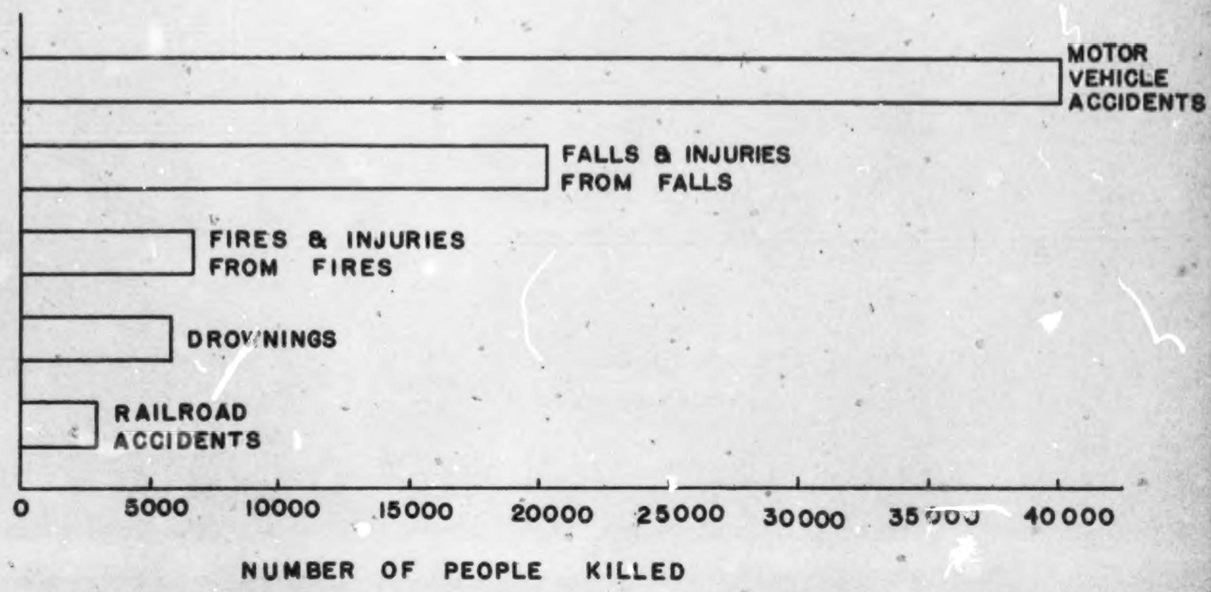
1. 3,260,000 × \$3.25 = \$10,595,000

2. 1959

3. Between 1958 and 1959. Table shows increase is 368,000.

4.

Accidental Deaths, 1956

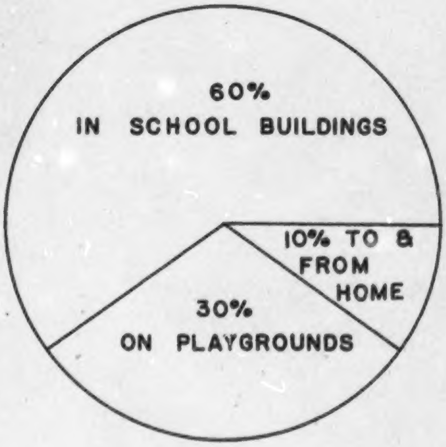


Exercises 11-4

Do not have all the pupils draw all four of the circle graphs in these exercises. Part of the pupils may do 1 and 3 while others are doing 2 and 4. If chalkboard protractors are available, the graphs might be drawn on the chalkboard.

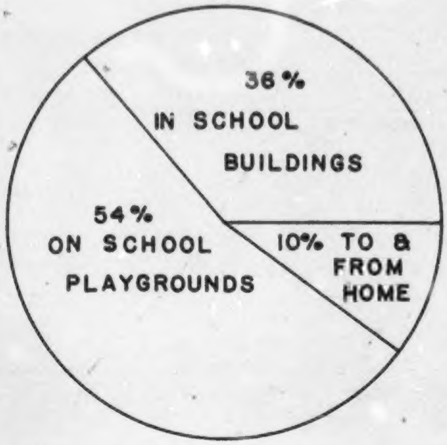
1\.

School-Related Accidents, 1949



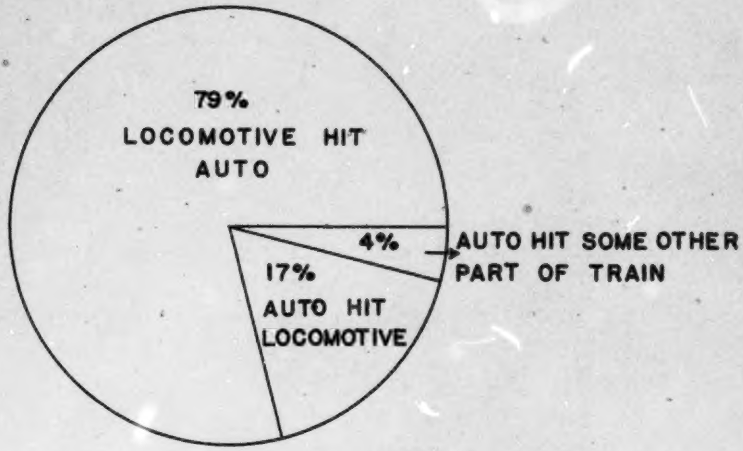
2\.

School-Related Accidents, 1956



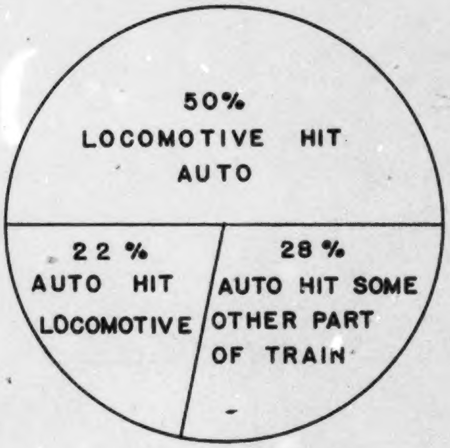
3. 1956 Railroad-Motor Vehicle Accidents in Daytime

PERCENT	NUMBER OF DEGREES
79	284.4
17	61.2
4	14.4



4. 1956 Railroad-Motor Vehicle Accidents at Night

PERCENT	NUMBER OF DEGREES
50	180
22	79.2
28	100.8



Exercises 11-5a

1. Mode: 85

2. (a) Arithmetic mean: $\frac{913}{11} = 83$

(b) Median: 85 (c) Range: 25

3. (a) Mean: \$6100

   (b) 3

   (c) 7

   (d) no

   (e) If there is an even number 2n of items (as there are 10 in this problem) the median is taken to be the average between the nth and (n + 1)th items.

$$\frac{\$5000 + \$5500}{2} = \$5250$$

   (f) Median is better than mean, since the mean gives the impression that the salaries are higher than they are. The mean is affected by the large salary of 12,500, but the median is not.

4. (a) Mean: 56.5

   (b) Median: 49

   (c) Range: 38

Answers to Exercises 11-5b

1. In making tables of ages of pupils in each grade.

   In tabulating data on distances from school, etc.

2. Median: between 15 and 17.

3.

Temperature	Frequency (No. in temperature range)
50 - 54	1
55 - 59	1
60 - 64	2
65 - 69	2
70 - 74	7
75 - 79	3
80 - 84	3
85 - 89	1
90 - 94	1
Total	21

$\frac{1}{2} \cdot 21 = 10\frac{1}{2}$ Consequently the middle number will be the 11th one. There must be ten temperatures below the median temperature and ten above. In order to find the temperature which corresponds to 11, we count from the top of the frequency column. To obtain 11 we must use 5 of the frequency 7. Hence the median is nearer 74 than 70. This may be checked by counting from the bottom of the table.

Answers to Exercises 11-5b

1. In 1953 and 1954.

2. 1951

3. $\frac{122.0}{10} = 12.2$

4. Average or mean is 83.4

   Deviations from the mean are +1.6, -1.4, +4.6, -7.4, +6.6, +.6, -3.4, -1.4, +.6, -.4.

   Average of these deviations from the mean is 2.8.

   This is called the average deviation from the mean.

5. Mean is 84.

   Deviations are +10, 0, -16, -10, +14, -14, +12, 0, -8, +12.

   Average deviation from the mean is 9.6.

*6. Suppose the guessed average is 45. Then the deviations of the scores are -5, -2, -1 and +2, +3, +4, +6.

The sum of the deviations for scores less than 45 is 8. The sum of the deviations for scores greater than 45 is 15.

$\frac{15 - 8}{7} = \frac{7}{7} = 1$. Since the sum of the deviations for the scores greater than 45 is more than the sum of the deviations for the scores smaller than 45 we add 1 to the guessed mean to get the correct mean, 45 + 1 = 46.

Suggested Test Items

Gross Debt of the United States for the Years 1861 to 1955 (Five-year Periods)

(a) Period	(b) Millions of Dollars	(c) Billions of Dollars	(d) Deviations from Mean
1861 - 65	2,678	3	
1866 - 70	2,436	2	
1871 - 75	2,156	2	
1876 - 80	2,091	2	
1881 - 85	1,579	2	
1886 - 90	1,122	1	
1891 - 95	1,097	1	
1896 - 1900	1,263	1	
1901 - 05	1,132	1	
1906 - 10	1,147	1	
1911 - 15	1,191	1	

Period	Millions of Dollars	Billions of Dollars	Deviations from Mean
1916 - 20	24,299	24	
1921 - 25	20,516	21	
1926 - 30	16,185	16	
1931 - 35	28,701	29	
1936 - 40	42,968	43	
1941 - 45	258,682	259	
1946 - 50	257,357	257	
1951 - 55	274,374	274	

1. (a) What is the mode of the set of figures in column (c)? In other words, which gross debt appears most often?

(b) Does this number tell you very much about the total debt picture of the United States Treasury since 1861?

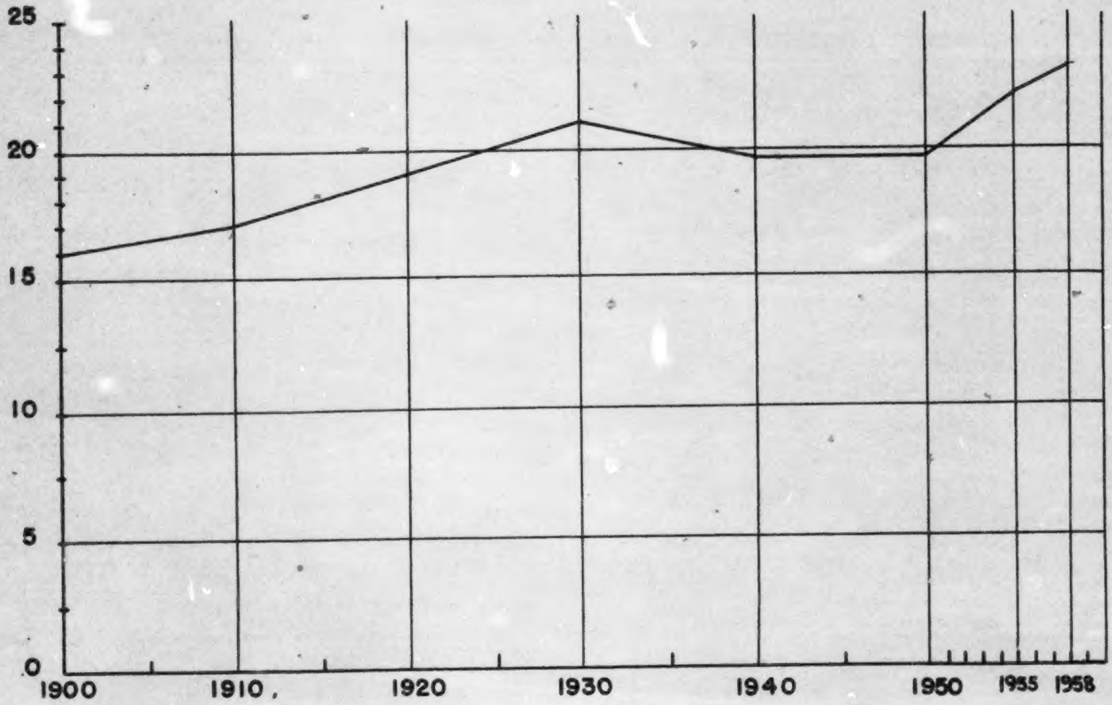
2. (a) Find the median of the set of figures in column (c) which will give you the median national debt from 1861 to 1955. This means that you will need to find the middle figure between 1 and 274. In order to do this, list the numbers from column (c) in order, and find the middle one on the list by counting up from the bottom or down from the top.

(b) Does the median tell you much about the total debt over this period of years?

Do you see that it is necessary to find the mean in order to take into consideration the very large numbers in the table?

3. Find the mean of the set of numbers in column (c) to the nearest whole number.
4. Find the deviations from the mean and then the average deviation to the nearest whole number.
5. Explain the large deviations above the mean.

Public School Enrollments in United States
Kindergarten and Grades 1--8
(Enrollment is given in millions for the years
shown on the horizontal base line.)



6. (a) What kind of graph is the graph in this problem?

   (b) What was the enrollment in 1910? 1920? 1930? 1958?

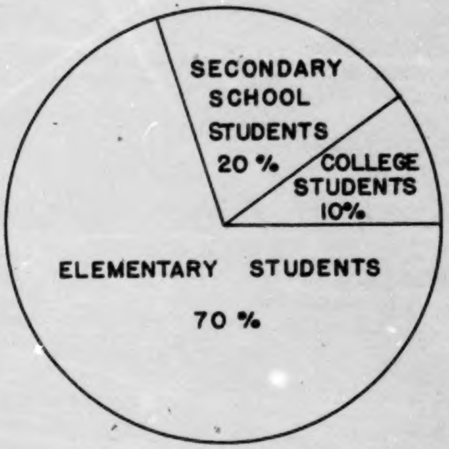
   (c) In which two years was the enrollment the same?

   (d) Estimate the enrollment in 1925.

   (e) In which ten-year period did the enrollment decrease?

   (f) If the graph continues in a straight line for the years 1955, 1958, and 1960, what would be the predicted enrollment in 1960?

School Enrollment
Year: 1950



7. (a) What fractional part of the total enrollment is in elementary school? in secondary school? in college?

   (b) Account for such a large group of children enrolled in elementary school?

   (c) If you were asked to draw this circle graph, how many degrees would there be in each central angle?

(d) If there were about 29,000,000 students enrolled in school in 1950, how many were in the elementary grades? in college? in secondary school?

Answers to Suggested Test Items

1. (a) Mode = 1 (b) No.

2. (a) Median = 2 (b) No.

3. Mean = 49, to nearest whole number of billions.

4. Deviations from mean are -46, -47, -47, -47, -47, -48, -48, -48, -48, -48, -48, -25, -28, -33, -20, -6, +210, +208, and +225, respectively.

5. World War II; Defense Program since World War II.

6. (a) Broken line graph.
    (b) 17 million, 19 million, 21, million, 23 million.
    (c) 1940, 1950.
    (d) About 20 million.
    (e) 1930 - 1940.
    (f) 24 million.

7. (a) $\frac{7}{10}$, $\frac{1}{5}$, $\frac{1}{10}$.
    (b) Compulsory school attendance and increase in birth rate.
    (c) 36°, 72°, 252°.
    (d) 20,300,000; 2,900,000; 5,800,000.

CHAPTER 12

MATHEMATICAL SYSTEMS

In this chapter, it is particularly important that teachers have clearly in mind both the *objectives* of the chapter and the suggested *method of approach* to be used with it.

The *main objective* is to lead the students to achieve some *appreciation of the nature of mathematical systems*. It is neither intended nor desirable that the children memorize the various tables introduced here or drill for mastery of the operations introduced here.

It is especially important that the teacher read this chapter through before planning his presentation and give considerable thought both to how to present some introductory motivation and even more to how to lead the students to *discover* the various relationships and properties of the chapter for themselves in advance of the reading of the text. The text itself attempts to suggest problems and processes for doing this as does this teacher's guide. However, these can be effective only if carefully planned for by the teachers. The process of discovering, of *perceiving for one's self* is a vital step in achieving our major objective: an appreciation of the nature of some types of mathematical systems. This is close to an appreciation of the nature of modern mathematics and of the work of mathematicians.

One of the most important activities of modern mathematicians is the search for common attributes or properties often found in apparently diverse situations or systems. Sometimes these common elements are deliberately built into new systems which are constructed as generalizations or abstractions of old systems as when the number system is extended from the system of counting numbers to the whole numbers, to the rational numbers, etc., etc. Sometimes these common elements are observed in systems less clearly related at first glance as when the changes of position of a rectangle into itself are conceived of as forming an algebraic system with a "multiplication" table which is discussed in the chapter.

Frequently, the systems developed out of the intellectual curiosity of mathematicians and their search for patterns in diverse abstract situations have been exactly the tools needed and seized upon by scientists in their attack on the problems of our physical world. The theory of groups, which actually has as its logical beginnings the properties discussed in this chapter, had its chronological beginnings in the early 19th century in problems relating to the solution of equations. Matrices, some of which form groups and give further examples of the principles of this chapter, were invented largely by the Englishman Cayley a little later. Within our generation the German physicist Werner Heisenberg has used matrices in the formulation of the quantum mechanics which is highly important in modern physics. Analogous stories relate the development of radio by Marconi to the differential equations of Maxwell, and point out that the outgrowths of Einstein's relativity theory owe much to his use of the tensor calculus developed by the Italian geometers Ricci and Levi-Civita. All of these stories have the same theme that both mathematicians and scientists are always seeking unifying principles or patterns. Frequently mathematics developed solely for the intrinsic interest of its properties and structure was later found to fit the needs of science, but for both science and mathematics we need to develop students who can see and understand patterns and structure.

In this chapter we are studying mathematical systems involving sets of elements and binary operations. Such systems which have certain simple additional properties are called groups and their study is a major branch of so-called "modern algebra". We shall not use all of these technical terms. However, other substantial objectives incidental to the major concern of this chapter and appropriate for secondary school students are:

1. Increased understanding of the nature and occurrence of the commutative, associative, and distributive properties, as well as the concepts of closure, identity element, inverse of an element.
2. Increased understanding of the inverse of an operation and its relationship to inverse and identity elements.

Additional discussions of these ideas and problem materials may be found in the books listed in the bibliography in Section 12-9.

12-1. A New Addition

Several of the sections, including the first, discuss the properties of what is referred to as modular arithmetic. The face of a clock is used to illustrate modular addition. The following quote provides background for the basic notion of this idea:

"In number theory we are often concerned with properties which are true for a whole class of integers differing from each other by multiples of a certain integer. Take, for instance, the fact that the square of an odd integer when divided by 8 leaves 1 for a remainder. Here we have a property holding for all odd numbers; that is, for a class of numbers differing from each other by multiples of 2. As another example, we see that when the last digit of a number, in decimal notation, is 6, then the last digit of its square will also be 6. Thus, in this simple example, we deal again with a property shared by integers differing by a multiple of an integer; namely, 10.

"The consideration of properties holding for all integers differing from each other by a multiple of a certain integer leads in a natural way to the notion of congruence. Two integers a and b whose difference $a - b$ is divisible by a given number m (not 0) are said to be congruent for the modulus m or simply congruent modulo m. Gauss, who introduced the notion of congruence, proposed the notation

$$a \equiv b(\text{mod } m)$$

to designate the congruence of a and b modulo m"[1]

Some textbooks use the following definition:

If $a \equiv km + b$, then $a \equiv b \pmod{m}$. The $\equiv$ sign is read, "Is equivalent to" or "is congruent to".

We emphasize that there is no need for the pupil to become familiar with the terms used in the above discussion, including "modular arithmetic."

In solving problems using replacements, encourage the pupil to make a list of possible replacements first. For mod 5 the set of possible replacements would be {0, 1, 2, 3, 4}; for mod 8, the set would be {0, 1, 2, 3, 4, 5, 6, 7}. These are examples of finite systems.

For the teacher's information each element of the set can be considered as an equivalence class, thus, numbers are put in equivalence classes.

Without a doubt, some pupils will wonder why the symbol " ≡ " ("is equivalent to") is used instead of " = .". This is an excellent opportunity to point out that the = sign is used when we have two names for the same thing; thus $3 + 2 = 4 + 1$ since these are two names for the same number, five. In the case of modular arithmetic, when we say "Five is equivalent to one, mod 4," the "five" and the "one" are not names of the same thing, thus it is necessary to introduce another symbol to describe this relationship.

---

[1] Uspensky and Heaslet, Elementary Number Theory, McGraw-Hill, 1939, page 126.

Answers to Exercises 12-1

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. (a) 1
   (b) 2
   (c) 2
   (d) 4
   (e) 2
   *(f) mod 5
   *(g) mod 6 or mod 3
   *(h) mod 7

3. mod 3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

mod 5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(Encourage the pupils to look for patterns when they are making tables.)

4. (a) 1
   (b) 1
   (e) 1
   (d) 2
   (e) 4
   (f) 4
   (g) 1
   (h) 0

The pupils may use the tables made in Exercise 3 above or make sketches of clocks.

5. $23 = 5(4) + 3$

The hand will go around four times and it would stop at 3.

12-2. What is an Operation?

Skills and Understandings

1. To recognize a binary operation described by a table.
2. To recognize a binary operation described in words.
3. To find, from a table, the result of putting two elements together in a binary operation described by the table.
4. To find, by computation, the result of putting two elements together in a binary operation described in words.
5. To tell, from the table for a binary operation, whether or not the operation is commutative.
6. To know that: (a) In order to show that a binary operation *is* associative, it is necessary to show that an equation (e.g. $a * (b * c) = (a * b) * c$) holds for *every* triple of elements a, b, c. (b) In order to show that a binary operation *is not* associative, it is sufficient to find *one* triple of elements a, b, c for which the equation does not hold (e.g., $a * (b * c) \neq (a * b) * c$).

Teaching Suggestions

To be given a binary operation, we must be given a set of elements and a way of combining any two elements to get a definite thing. The "definite thing" may or may not belong to the original set of elements. The two elements we combine may be the same element taken twice. If the operation is given to us by a table, the set is composed of those elements which appear in the left-hand column and in the top row (the same elements must appear in both places). For example, the set for the operation of table (c) is {0, 1, 2, 3}; that for the operation of table (d) is {1, 2, 3}. In table (d), all the entries in the table belong to the set {1, 2, 3}; in table (c) many of the entries in the table do not belong to the set {0, 1, 2, 3}. This point is discussed more fully in the next section on closure.

Bring out by class discussion that the entries in the tables (the results of putting two elements together) could be anything at all. As later examples will show, they do not have to be numbers.

Practice on reading the tables. Stress that, in evaluating 1 □ 3, the "1" is to be found in the left column, and the "3" in the top row. Point out that 1 □ 3 = 5 and 3 □ 1 = 7, so it is necessary to be careful about the order in which elements are written.

Some examples for class discussion are given below.

Example 1. Set: The counting numbers.

Rule of Procedure: Given any two elements, take twice the first and add three times the second. This is an operation, but it is not commutative and it is not associative.

Example 2. Set: The counting numbers.

Rule of Procedure: Given any two elements, take twice one of them and add three times the other.

This rule does not define an operation since a "definite thing" is not always determined. For instance, in combining 2 and 3, we are allowed to form either $2 \cdot 2 + 3 \cdot 3 = 13$ or $2 \cdot 3 + 3 \cdot 2 = 12$. The result of an operation applied to two elements must be unique, that is, there can be one and only one answer. It could also be seen from a table that this rule does not describe an operation. The table would have more than one entry in some places (everywhere except on the diagonal from upper left to lower right).

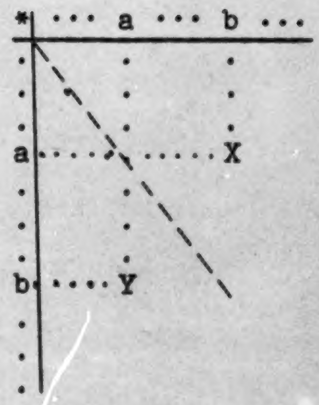
Example 3. Set: The whole numbers.

Rule of Procedure: Given any two elements, divide the first by the second.

This rule does not define an operation, since division by zero is impossible. The elements 2 and 0 cannot be put together in that order. Notice that, in the order 0 and 2, they can be combined (the result is zero). It could also be seen from a table that this rule does not describe an operation. The table would have some of the spaces blank (the column with "0" at the top would be blank).

Discussion of Exercises 12-2

2. In the schematic diagram at the right, a * b is to be entered at position X, and b * a is to be entered at position Y. These two positions are symmetrically located with respect to the diagonal from upper left to lower right. (The elements are arranged in the same order in the top row and left column.) If an operation is commutative, its table will be symmetric about this diagonal, and conversely.



3. Bring out by discussion that, to prove an operation is associative, requires testing every triple; one example would be sufficient to prove an operation is not associative. This logical point has been illustrated several times previously, and should be re-emphasized here. To prove associativity for an operation by examining all the cases is almost always a long process. Each student should check 2 or 3 cases and if all of them are satisfactory, the following statement can be made: "This operation appears to be associative, but we are not really sure."

4. In making a table for each of these operations, arrange the elements of the set in the same order in the top row and

left column. Compute and fill in as many entries in the table as needed to see the pattern. Associativity can be decided from known properties of the counting numbers.

(a) This operation is not completely described. If the two given numbers are equal, there is no smaller one, of course. Bring out, by class discussion, that, if the two numbers are the same, the result of the operation should be defined as that same number.

(b) and (c) Same as (a) above. Consider the case where the two numbers are the same and decide that the result of the operation will be that number.

5. and 6. The students will need help in beginning these problems. The successive steps are as follows: (1) Choose a set (each pupil may have a different set, but it is better not to have too many elements in each set so the problem will not be too long). Suppose the set {1, 2, 3} is chosen. (2) Make the framework for a table as shown at right. The elements in the set which was chosen in (1) will appear in the left column and top row. Arrange them in the same order. (3) Choose a symbol, such as *, for the operation and put it in the upper left-hand corner of the framework; make up a name to go with it, such as "star". (4) Fill in the table. Emphasize that

*	1	2	3
1			
2			
3			

the names of any objects whatever may be placed in the body of the table - it is not necessary that these objects be elements of the set chosen in (1). If the operation is to be commutative, the table must be symmetric about the diagonal from upper left to lower right. If the operation is not to be commutative, the table must not be symmetric.

7. Point out that, if a square array is used as in our other tables, then only the diagonal will be filled in, since only one element is required to perform this unary operation. (We cannot put two elements together.) A shorter way to write the information is to arrange the elements and the corresponding results of the operation in two rows or columns. Usually some symbol (such as "x") is used to denote an element of the set and a different symbol (such as "y" or, in this case, "x^3") denotes the corresponding result of the operation. The table is given in two columns in the answers. It could also be written in two rows as shown below.

x	0	1	2	3	4	5	6	7	8	9	10
x^3	0	1	8	27	64	125	216	343	512	729	1000

Answers to Exercises 12-2

1. (a) 1
   (b) 6
   (c) 8
   (d) 7
   (e) 2
   (f) 3
   (g) 1
   (h) 1
   (i) 8
   (j) Not possible; 1 □ 2 = 4 and 1 □ 4 is not defined since 4 does not appear in the rop row.

2. (a), (b), (d). The table must be symmetric about the diagonal from upper left to lower right. See discussion, Exercises 12-2.

3. There is no short cut method; to prove associativity each triple of elements must be combined in the two ways and the corresponding results must be equal. The operations of Tables (a) and (d) are associative; those of Tables (b) and (c) are not. See discussion, Exercises 12-2.

4. See discussion, Exercises 12-2. The operation symbols are omitted in the following tables:

(a)

	26	27	28	...	74
26	26	26	26	...	26
27	26	27	27	...	27
28	26	27	28	...	28
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
74	26	27	28	...	74

Commutative: Yes

Associative: Yes

(b)

	501	502	503	...	535
501	501	502	503	...	535
502	502	502	503	...	535
503	503	503	503	...	535
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
535	535	535	535	...	535

Commutative: Yes

Associative: Yes

(c)

·	2	3	5	7	11	···
2	2	3	5	7	11	···
3	3	3	5	7	11	···
5	5	5	5	7	11	···
7	7	7	7	7	11	···
11	11	11	11	11	11	···
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(d)

	40	42	44	···	60
40	40	40	40	...	40
42	42	42	42	...	42
44	44	44	44	...	44
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
60	60	60	60	...	60

Commutative: No

Associative: Yes

(e)

	1	2	3	···	49
1	3	4	5	···	51
2	5	6	7	···	53
3	7	8	9	···	54
.	.	.	.	···	.
.	.	.	.	···	.
49	99	100	101	···	147

Commutative: No

Associative: No

(Try the triple
1,2,3.)

(f)

	1	2	3	4	5	6	···
1	1	1	1	1	1	1	···
2	1	2	1	2	1	2	···
3	1	1	3	1	1	3	···
4	1	2	1	4	1	2	···
5	1	1	1	1	5	1	···
6	1	2	3	2	1	6	···
.	.	.	.	.	.	.	···
.	.	.	.	.	.	.	···
.	.	.	.	.	.	.	···

Commutative: Yes

Associative: Yes

(g)

	1	2	3	4	5	...
1	1	2	3	4	5	...
2	2	2	6	4	10	...
3	3	6	3	12	15	...
4	4	4	12	4	20	...
5	5	10	15	20	5	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(h)

	1	2	3	4	...
1	1	1	1	1	...
2	2	4	8	16	...
3	3	9	27	81	...
4	4	16	64	256	...
.	.	.	.	.	...
.	.	.	.	.	...
.	.	.	.	.	...

Commutative: No

Associative: No

(Try the triple 2, 1, 3;

$(2^1)^3 = 8 \neq 2 = 2^{(1^3)}$

5. Many answers are possible, of course. The only requirement is that the table be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation). See discussion, Exercises 12-2.

*	1	2	3
1	X	Y	Z
2	Y	P	Q
3	Z	Q	R

6. Many answers are possible, of course. The only requirement is that the table must not be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation). See discussion, Exercises 12-2.

*	1	2	3
1	X	P	Z
2	P	R	Y
3	Q	P	Q

*7.

x	x^3
0	0
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729
10	1000

See discussion, Exercises 12-2.

12-3. Closure

Skills and Understandings

1. To recognize, from the table describing a binary operation, whether or not a set is closed under the operation.
2. To find whether or not a set is closed under a binary operation described in words.

Teaching Suggestions

The concept of closure has been discussed in Chapters 3 and 6 in connection with the usual arithmetic operations. The discussion here should prepare the pupil for consideration of more general systems where the elements may not be numbers.

Bring out, by class discussion, that closure involves two things. (1) It must be possible to put any two (not necessarily different) elements of the set together and (2) the result obtained must always be an element of the set. Material for class discussion is provided by the various parts of Problem 4 of Exercises 12-2.

As with associativity (see discussion of Problem 3, Exercises 12-2), to prove a set *is* closed under an operation, all cases must be considered; a single counterexample would prove that the set *is not* closed under the operation.

Discussion of Exercises 12-3

1. Each table determines a set (the set of elements in the

left column and top row), and completely describes the corresponding operation. For a set to be closed under the corresponding operation, each entry in the body of the table must be an element of the set. In Tables (a) and (d), this is true; in (b) and (c), it is not.

*3. From the definition of commutativity in Section 12-2, it must be possible to put any two elements of the set together in either order and the same result must be obtained, but the result of the operation is not required to be an element of the set. In fact, Table (b) of Section 12-2 gives an example of a commutative operation, and the set on which the operation is defined is not closed under the operation.

*4. From the definition of associativity in Section 12-2, it must be possible to put any three elements of the set together in the two ways specified and the same result must be obtained. This means that the set on which the operation is defined must be closed under the operation since, if we can combine a, b, c as $(a + b) + c$, then certainly $a + b$ must be an element of the set on which the operation is defined. That is, the set is closed under the operation.

5 and 6. The pupils may need help in beginning these problems. The set of elements has been chosen, but each pupil should

choose a symbol for his operation and fill in the entries in the table. See discussion of Problems 5 and 6, Exercises 12-2.

Answers to Exercises 12-3

1. The sets of (a) and (d) are closed under the corresponding operations (all the entries in the table appear in the left column and in the top row); those of (b) and (c) are not closed (some entries in tables (b) and (c) do not appear in the left column and in the top row). See discussion, Exercises 12-3.

2. (a) Closed
   (b) Closed
   (c) Closed
   (d) Not closed
   (e) Closed
   (f) Not closed: 15 - 35 cannot be performed
   (g) Closed
   (h) Closed
   (i) Not closed: 3 + 7 is not a prime
   *(j) Not closed: 3 + 3 = 11 (base 5)

*3. No; see discussion, Exercises 12-3.

*4. Yes; see discussion, Exercises 12-3.

*5. Many answers are possible, of course. The only requirement is that each entry in the table belong to the set {0, 43, 100}. (And that each place in the table be filled uniquely so the table does describe an operation.) See discussion, Exercises 12-3.

*	0	43	100
0	0	0	43
43	43	0	43
100	0	43	0

*6. Many answers are possible, of course. The only requirement is that at least one entry in the table must not be an element of the set {0, 43, 100}. (And that each place in the table be filled uniquely so the table does describe an operation.) See discussion, Exercises 12-3.

*	0	43	100
0	0	0	43
43	43	1	0
100	2	0	43

12-4. Identity Element; Inverse of an Element

Skills and Understandings

1. To determine from a table whether there is an identity element for the operation, and if so, what it is.
2. To realize that an element cannot have an inverse unless there is an identity element.
3. To determine from a table which elements have inverses.
4. To find the inverse of an element, if the element has an inverse.

Teaching Suggestions

Let the students experiment with several tables finding identity elements and inverses of elements. Try to lead them to discover that there is an identity element for an operation if, in the table, (1) there is a column exactly like the left column, and (2) there is a row exactly like the top row. The element associated with both will be the same, and will be the identity.

*	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

In the figure the last row and the last column fit the above conditions. 5 is the identity element.

*	3	4	5	1	2
1	4	5	1	2	3
2	5	1	2	3	4
4	2	3	4	5	1
5	3	4	5	1	2
3	1	2	3	4	5

The third column and the fourth row fit the conditions. 5 is the identity.

Lead the students to discover that an element has an inverse if the identity appears in the same relative position in the row as in the column associated with this element when the top row and left column are in the same order.

For example: In the first table the second element in the third row and the third column is the identity element 5. This means that 3 has an inverse. Since 3 was associated with 2 both times to get the identity 5, then 2 and 3 must be inverses. The pairs 1 and 4 and 5 and 5 are seen to be inverses in a similar way.

Notice that the second table has the same elements and the same operation as the first, but that the order of the elements in the left column is different from the top row. It is not possible now to use our usual check of symmetry about the diagonal for commutativity. The method of finding the inverse of an element discussed above does not work out either.

The above may be pointed out to the students if you wish. None of the tables in the chapter will have its top row and left column in different order.

Answers to Exercises 12-4

1. (a) In table (a), the identity is 5.

In table (d), the identity is 2.

(b) In table (a), the inverse of 1 is 4; of 2 is 3; of 5 is 5.

In table (b), no element has an inverse.

In table (c), no element has an inverse.

In table (d), the inverse of 1 is 3; of 2 is 2.

Each member of the sets for tables (a) and (d) has an inverse. The operations described by tables (b) and (c) do not have identities so no inverses can exist.

2. (a)

Operation	Identity
(a)	74
(b)	501
(c)	2
(d)	None
(e)	None

(f) None

(g) 1

(h) None

(b) The only inverses are those listed below.

(a) 7^4 is the inverse of 7^4.

(b) 501 is the inverse of 501.

(c) 2 is the inverse of 2.

(g) 1 is the inverse of 1.

(c) None.

3. No; if there are two identities (P and Q) for a given operation, then consider the result when P is combined with Q. Since Q is an identity, the result must be P. But since P is also an identity, the result must be Q. Thus, P and Q must be the same element since each equals the result of combining P and Q.

12-5. What is a Mathematical System?

Skills and Understandings

1. To recognize a mathemtical system described by a table or in words.
2. To perform the operation of a mathematical system; recognize identity elements, inverses of elements and properties of the operations.

Teaching Suggestions

In Section 12-2, it was pointed out that a table describes a set and an operation defined on that set. Thus, a table really describes a mathematical system, and not merely an operation. Illustrate by discussing tables (a) - (d) of Section 12-2, and showing that each table does describe a mathematical system (a set and one or more operations defined on that set--in each case, it will be one operation). Point out that the systems of Problem 4, parts (b) and (c), Exercises 12-2 are different, since two different sets are involved.

In Example 1, part (c) (egg-timer arithmetic), remind the pupils of the symmetry test for commutativity discovered in Problem 2 of Exercises 12-2. The table for egg-timer arithmetic is symmetric, so the operation is commutative.

Have the class decide on a word for the operation in table (c) of this section. ("Twiddle" is sometimes used.)

Answers to Exercises 12-5

1. Each one of Tables (a), (b), (c) describes a mathematical system.

   For Table (a), the set is {A, B}; the operation is o.

   For Table (b), the set is {P, Q, R, S}; the operation is *.

   For Table (c), the set is {△, □, ○, \ }; the operation is $\sim$.

2. (a) A

(b) ○

(c) ○

(d) B

(e) Q

(f) S

(g) P

(h) \

(i) △

(j) B

(k) A

(l) R

3. The operation o is not commutative, since Table (a) is not symmetric.

The operations * and $\sim$ are both commutative, since both of Tables (b) and (c) are symmetric.

4. There is no identity element for the operation o. There is no element e, such that both of the equations A o e = A and B o e = B are correct.

The element R is the identity element for the operation *. The row of Table (b) with "R" in the left column is the same as the top row, and the column with "R" at the top is the same as the left column.

The element △ is the identity element for the operation $\sim$. The first row and column of Table (c) are the same as the top row and left column respectively.

5. (a) S

(b) S

(c) R

(d) R

(e) Q

(f) Q

(g) \

(h) \

(i) \

(j) \

6. Each of the operations * and seems to be associative since, in each of the cases we have tried, the corresponding expressions are equal. To prove the operations are associative, we would have to examine all cases and show that the corresponding expressions are equal. To prove an operation is not associative, a person would have to find one example where the corresponding expressions are not equal.

7. BRAINBUSTER. (a) The element 2 cannot be combined with 2 by the operation * (that is, 2 * 2 is not defined).

   (b) 2 * 1 is not uniquely defined. Many results are possible when 2 and 1 are combined.

   (c) The set given by this table is {1, 2, 3, 4}. But it is not possible to combine every pair of elements (e.g. 3 and 3). We do not have an operation _defined on the set_.

12-6. Mathematical Systems Without Numbers

Skills and Understandings

1. To recognize a mathematical system when it is described in words.
2. For systems without numbers: To recognize the element of the set, to recognize the operation; to recognize an identity element; to recognize the inverse of an element.

Teaching Suggestions

Each pupil should have his own rectangle to manipulate, such as, a 3" × 5" card. Do not use square cards. Be sure that each pupil labels his rectangle correctly so that comparisons between different pupils are possible. Check especially, that each corner of the card is labeled with the same letter on both sides. Stress that the card is used only to represent a geometric figure--the interior of a rectangle. Some of the geometric concepts of Chapters 4 and 9 should be reviewed here.

It cannot be repeated too often that the changes of position of a rectangle are the elements of the set in the mathematical system discussed in this section. One of these changes is something that is "done"; that is, it is a physical activity, but it is an element of the set--it is not the operation of the system. The operation of the system is much more elusive. Any operation defined on the set must be a way of combining any two of these physical activities (changes) to get a definite thing. The particular operation we have chosen combines two of these changes by doing the first one and then the other. The result (definite thing) obtained is one of the changes, but the operation is the way of combining them, that is: First do ..., and then do ....

For ease in grading pupils' written work it is essential that all students use the same notation in exercises 5 and 6*. One possible notation is described in the answers.

Discussion of Exercises 12-6

3. In proving associativity, "all cases" must be considered. There is one case for each triple of (not necessarily different) elements of the set on which the operation is defined. For the operation ANTH, there are 4 elements in the set, so there will be $4 \cdot 4 \cdot 4 = 64$ triples; that is, 64 cases must be considered to prove the associative property.

5. and *6. For ease in grading written work it is essential that all students use the same notation in these exercises. One possible notation is described in the answers.

Answers to Exercises 12-6

1.

ANTH	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

2. (a) V
(b) V
(c) V
(d) V
(e) I
(f) I
(g) I
(h) I
(i) I

3. (a) Yes

(b) Yes

(c) Yes, the operation is associative. A proof would require that 64 cases be checked. Each pupil should check two or three; do not attempt to check all cases. See discussion, Exercises 12-6.

(d) Yes. I is the identity.

(e) Yes. Each element is its own inverse.

4. (a)

ANTH	I	F
I	I	F
F	F	I

(b) Yes

(c) Yes

(d) Yes. All cases can be checked (there are 8 cases in all). See discussion of Problem 3, Exercises 12-6.

(e) Yes. I is the identity element.

(f) Yes. Each element is its own inverse.

5.

ANTH	I	R	S	T	U	V
I	I	R	S	T	U	V
R	R	S	I	U	V	T
S	S	I	R	V	T	U
T	T	V	U	I	S	R
U	U	T	V	R	I	S
V	V	U	T	S	R	I

The operation is not commutative (R ANTH T $\neq$ T ANTH R)

I is the identity element. Each of I, T, U, V is its own inverse element; R and S are inverses of each other.

*6. Notation:

I: Leave the square in place.

R_1: Rotate clockwise $\frac{1}{4}$ of the way around.

R_2: Rotate clockwise $\frac{1}{2}$ of the way around.

R_3: Rotate clockwise $\frac{3}{4}$ of the way around

H: Flip the square over, using a horizontal axis.

D_1: Flip the square over, using an axis from upper left to lower right.

D_2: Flip the square over, using an axis from lower left to upper right.

ANTH	I	R_1	R_2	R_3	H	V	D_1	D_2
I	I	R_1	R_2	R_3	H	V	D_1	D_2
R_1	R_1	R_2	R_3	I	D_2	D_1	H	V
R_2	R_2	R_3	I	R_1	V	H	D_2	D_1
R_3	R_3	I	R_1	R_2	D_1	D_2	V	H
H	H	D_1	V	D_2	I	R_2	R_1	R_3
V	V	D_2	H	D_1	R_2	I	R_3	R_1
D_1	D_1	V	D_2	H	R_3	R_1	I	R_2
D_2	D_2	H	D_1	V	R_1	R_3	R_2	I

I is the identity change. The operation is not commutative (R_1 ANTH H $\neq$ H ANTH R_1).

12-7. The Counting Numbers and the Whole Numbers

This section has problems which lead the pupils to conclude that the counting numbers and the whole numbers each form a mathematical system. It is pointed out that the distributive property with which the pupil is familiar comes from the abstract discussion of this property. The pupils should not be expected to duplicate the abstract definition.

One of the objectives of the section is to serve as a way to pull together the concept of systems.

Some of the sets of numbers considered in ordinary arithmetic are: the rational numbers, the whole numbers, the counting numbers, the even numbers, etc.

Discussion of Exercises 12-7

4. One possible model of the mathematical system in this exercise is as follows: Let $A = \{1, 2\}$ $B = \{1, 2, 3\}$, $C = \{1, 2, 4\}$, $D = \{1, 2, 3, 4\}$. Then, from the tables in the problem, the operation $*$ is intersection and the operation $\circ$ is union. Each of these operations distributes over the other.

Answers to Exercises 12-7

1. (a) Since the sum of two counting numbers is always another counting number and the product of two counting numbers is always a counting number, the set is

closed under addition and multiplication.

(b) Both the commutative property and the associative property hold for addition and multiplication.

Examples: Commutative: $2 + 3 = 3 + 2$;

$4 \times 6 = 6 \times 4$

Associative: $3 + (4 + 7) = (3 = 4) + 7$;

$3 \times (6 \times 8) = (3 \times 6) \times 8$.

(c) There is no identity element for addition.

The identity element for multiplication is 1; for every counting number n, $n \cdot 1 = n = 1 \cdot n$.

(d) The counting numbers are not closed under subtraction or division.

2. (a) The set of whole numbers is closed under addition and multiplication.

(b) Both operations are commutative and associative.

(c) There is an identity element for addition. It is zero; for any whole number n, $n + 0 = n = 0 + n$.

The number 1 is the identity element for multiplication.

The answers are the same as for 1 (a), (b), (c) except that there is an identity element for addition in the whole number system and not in the counting number system.

3. (a) Three examples are: $2(3 + 4) = (2 \cdot 3) + (2 \cdot 4)$; $5(7 + 10) = (5 \cdot 7) + (5 \cdot 10)$; $1(1 + 1) = (1 \cdot 1) + (1 \cdot 1)$.

   (b) Addition does not distribute over multiplication; for example, $2 + (3 \cdot 4) = 14 \neq 30 = (2 + 3)(2 + 4)$.

4. See discussion, Exercises 12-7.

   (a) Yes, here are 3 illustrations that * distributes over o:

   $A * (B \circ C) = A = (A * B) \circ (A * C)$

   $B * (B \circ B) = B = (B * B) \circ (B * B)$

   $C * (B \circ D) = C = (C * B) \circ (C * D)$

   (b) Yes, here are 3 illustrations that o distributes over *.

   $A \circ (B * C) = A = (A \circ B) * (A \circ C)$

   $B \circ (B * B) = B = (B \circ B) * (B \circ B)$

   $C \circ (B * D) = D = (C \circ B) * (C \circ D)$.

5. (a) Closed; Commutative; Associative; 1 is the identity; only the number 1 has an inverse.

   (b) Closed; Commutative; Associative; no identity; no inverses.

   (c) Closed; Commutative; Associative; 0 is the identity; only the number 0 has an inverse.

   (d) Closed; Commutative; Associative; no identity; no inverses.

(e) Closed; Commutative; Associative; 0 is the identity; only the number 0 has an inverse.

(f) Not closed; Commutative; not associative; no identity; no inverses.

6. (a) Both sets are closed under the operations. Both operations are commutative and associative. Both systems involve the same set.

(b) The system 5(a) has an identity and 5(b) does not. Also, the sets are different in these two systems.

*7. Many results are possible, of course.

*8. (a) Yes. We are asked to consider the two expressions $a * (b \circ c)$ and $(a * b) \circ (a * c)$, and find whether or not they are always equal. For example, using

$a = 8,\ b = 12,\ c = 15,$

$8 * (12 \circ 15) = 8 * 60 = 4.$

$(8 * 12) \circ (8 * 15) = 4 \circ 1 = 4.$

(b) Yes. We are asked to consider the two expressions $a \circ (b * c)$ and $(a \circ b) * (a \circ c)$, and find whether or not they are always equal. For example, using

$a = 8,\ b = 12,\ c = 15,$

$8 \circ (12 * 15) = 8 \circ 3 = 24$

$(8 \circ 12) * (8 \circ 15) = 24 * 120 = 24.$

12-8. Modular Arithmetic

In this section, the number line is used to provide a picture of how equivalence classes of whole numbers can be developed. At this time it may be wise to re-read the first paragraphs of Section 12-1. We use the term "multiple" to mean "multiple by a whole number".

Problems which may be used for motivation to explain the meaning of modular systems include the ordinary 12-hour clock, the days of the week, and the months of the year. For example, "Today is Tuesday; what day will it be six days from now?" Answer: Monday; this is mod 7. "It is 4:30 o'clock. What time will it be 10 hours from now?" Answer: 2:30; this is mod 12.

Modular arithmetic may be thought of as a mathematical system with two operations. Section 1 discussed modular addition and Section 8 discusses modular multiplication. The two operations together allow us to use the distributive property; thus, the whole numbers form a system under modular addition and multiplication. In modular arithmetic only a finite number of symbols is needed because infinitely many whole numbers are represented by each symbol.

Other interesting highlights are:

A product of non-zero factors may be zero in some systems.

There may be many replacements for x in a number sentence to make it true.

Answers to Exercises 12-8

1. Mod 5

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Mod 8

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(Encourage the pupils to look for patterns and to use what they have previously learned about systems to make the tables.)

2. (a) Mod 5: Yes; mod 8: Yes
   (b) Mod 5: Yes; mod 8: Yes
   (c) Mod 5: Yes; mod 8: Yes
   (d) Mod 5: 1; mod 8: 1
   (e) Mod 5: 1 and 4 are their own inverses; 2 and 3 are inverses of each other; 0 has no inverse.
   Mod 8: Only 1 has an inverse; its inverse is 1.
   (f) Mod 5: Yes; mod 8: No, $2 \times 4 \equiv 0 \pmod 8$.

3. (a) 3 (c) 6, 8, 12, 24
   (b) 2 (d) 4, 8

4. (a) 2 (d) 0

(b) 0 (e) 4 *(g) 1. Any power of 6 ends in 6.

(c) 5 (f) 1

5. (a) 4 (c) 1

(b) 2 (d) 3

6. (a) Yes; a(b + c) = ab + ac so surely a(b + c) and ab + ac are in the same class mod 5. That is, a(b + c) ≡ ab + ac (mod 5).

(b) No. Try the triple (1, 1, 1).

*7. (a) 6 (e) 0

(b) 3, 7 (f) 0

(c) 0, 4 (g) 9

(d) 2 (h) Not defined in this system.

8. (a) 4; What number added to 3 gives 7?

(b) 4

(c) 7

*(d) 7

9.

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

The set is closed under subtraction mod 5.

10. (a) 3, 8, 13 and others (add 5)

(b) 3, 7, 11 and others (add 4)

(c) 0, 5 and so on.

(d) Any even number

(e) 3, or any odd number greater than 3.

(f) 1, 3, 5 and so on.

11. (d) Any even number

(f) 1, 3, 5, 11, 13 and so on.

Answers to Sample Review Questions

1. 2
2. 4
3. {6, 8, 10}
4. $\frac{2}{5}$
5. 4
6. $\frac{9}{2}$, or $4\frac{1}{2}$, $\frac{18}{4}$, $\frac{27}{6}$, etc.
7. Check to see if ad = bc.
8. $\frac{1}{4}$ inch
9. 10.5
10. $\left(\frac{7}{8}\right)^6$ or $\frac{7^6}{8^6}$
11. 58°
12. Mode

Sample Test Questions

Part I. True - False

T 1. Operations can be described by tables.

T 2. A symbol can be made to mean anything providing we define it.

F 3. The identity for multiplication in ordinary arithmetic is zero.

F 4. The identity for addition in ordinary arithmetic is one.

T 5. The additive inverse of 2 in the mod 4 system is 2.

T 6. In ordinary arithmetic, with the set composed of all the rational numbers except zero, the inverse of division is multiplication.

F 7. All mathematical systems are sets of numbers.

T 8. In mod 5 arithmetic, $0/3 \equiv 2 \pmod 5$

T 9. The set $\{0, 1, 2, 3\}$ is closed under subtraction mod 4.

Part II. Computation.

Find the sums:	Answers:
1. $(9 + 2) \pmod{12}$	11
2. $(5 + 4 + 3) \pmod 6$	0
Find the differences:	
3. $(5 - 2) \pmod 6$	3
4. $(3 - 5) \pmod 7$	5
Find the products:	
5. $[(3 + 7) \times 6] \pmod 9$	6
6. $3^2 \pmod 8$	1

Find the quotients:

	Answers
7. $\frac{2}{3}$ (mod 5)	4
8. $\frac{0}{7}$ (mod 11)	0

Part III. Multiple Choice

The table below describes a mathematical system. It is to be used in answering questions 1, 2, and 3 below.

o	A	B	C	D
A	C	D	A	B
B	D	A	B	C
C	A	B	C	D
D	B	C	D	A

1. Which one of the following statements is true? (Answers are starred).

   A. The set {A, B, C, D} is not closed with respect to the operation o.

   *B. The operation o is commutative.

   C. The operation o does not have an identity element.

   D. The operation o is not associative.

   E. None of the above.

2. The identity for the operation o is:

   A. D

   B. B

   *C. C

   D. Both A and B

E. None of the above.

3. In the mathematical system:

A. Only B has an inverse.

B. Only D has an inverse.

C. Only A and C have inverses.

D. None of the elements has an inverse.

*E. All the elements have inverses.

4. For what modulus m is $2 - 5 \equiv 4$ (mod m) true?

A. Mod 9

B. Mod 6

C. Mod 8

*D. Mod 7

E. None of the above.

5. For the system consisting of the set of odd numbers and the operation of multiplication:

A. The system is not closed.

B. The system is not commutative.

C. The system has no identity element.

*D. None of the above is correct.

E. All of the above are correct.

6. For the system consisting of the set of even whole numbers and the operation of addition:

A. The system is not closed.

*B. The system has an identity element.

C. The system has an inverse for addition for each element;

D. All of the above are correct;

E. None of the above is correct;

7. A mathematical system consists of several things. Which of the following is always necessary in a mathematical system?

A. Numbers

B. An identity element

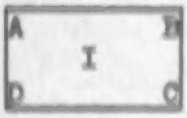
C. The commutative property

*D. One or more operations

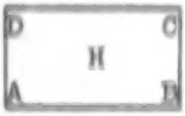
E. None of the above

Use the mathematical system as described below in answering questions 8, 9, and 10. The set of elements in our system is the set of changes of a rectangle.

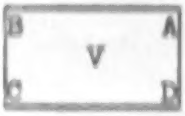
The elements are:



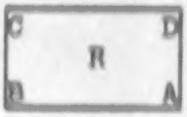
I means leave alone.



H means flip on the horizontal axis.



V means flip on the vertical axis



R means turn halfway around its center.

The following is an illustration of our operation *

V * H means do change V and then do change H.

Thus V * H = R.

8. H * H is:

A. H

*B. I

C. R

D. V

E. None of the above.

9. I * R is:

*A. R

B. V

C. I

D. H

E. R * H

10. (H * V) * V is:

A. I

B. V * V

*C. H * I

D. V

E. None of the above.

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CHAPTER 13

MATHEMATICS AT WORK IN SCIENCE

13-1. The Scientific Seesaw

This unit can be used in a mathematics class or a science class or it can be planned as a joint project. The chapter is rather short and probably can be completed in 4 or 5 days. It might well follow at any time after Chapter I if special instruction is given in plotting points and drawing a graph.

Much of the science that is taught in the schools today is descriptive science, and it omits or neglects the studies of a quantitative nature by which one learns to carry out the main steps of a scientific experiment, namely:

(a) The scientist observes what happens and collects and studies the quantitative data.

(b) To explain the facts he has observed, he states his hunch as a hypothesis which expresses the pattern he sees in the data.

(c) He makes predictions, telling what will be seen if certain other observations in the experiment are made.

(d) He goes back to the equipment and tests his prediction. If it works in a reasonable number of cases, he can state his discovery as a theory or principle. No amount of tests will "prove" the principle, but a rather large number of tests will show that the principle is very likely to be true. The principle is usually stated with mathematical terms and symbols.

The purposes of this chapter are:

(a) to illustrate by a simple experiment the typical inductive method of science;

(b) to give pupils experience in collecting mathematical data in an experiment in science;

(c) to help pupils see the importance of the quantitative aspect of the physical experiment;

(d) to furnish applications of some mathematical skills.

11-2. A Laboratory Experiment

The equipment is simple. If gram weights are not available, new pennies can be used, or new marbles from the same manufacturer. These are colorful and about right as units of weight, but they roll when they are dropped. A meter stick or yardstick is needed to serve as the balance rod or seesaw.

The law of the lever is one that can be discovered rather easily if measurements are carefully made; and yet it is fundamental in science. It is used in much equipment that we see around us such as crowbars, tongs, and scissors.

It is recommended that each pupil take part in the experiment. The teacher may find it profitable to divide the class into groups of four or five pupils so that each group can carry out its own experiment. A special committee of pupils should be appointed in advance to secure the necessary equipment and to prepare it for use.

Squares about 8 inches on the side may be cut from the thin plastic covers or bags used in packaging. The four corners may be tied together with sturdy thread which is then looped around the meter stick or yardstick. The stick may be suspended by the same kind of thread. To simplify reading distance measurements, the stick should be balanced at the center point before the bags are suspended. A thumbtack at the proper place will help balance the stick if it is out of balance in the beginning.

The problem of finding a support from which to suspend the meter stick (balance bar) may take some ingenuity to solve. It may be possible to borrow standards from the science department or to manufacture hooks by using coat hangers. As a last resort some teachers have used another stick resting across the backs of two chairs as a support for the balance.

Be sure to give complete and clear instructions in advance so that each pupil understands what is to be done.

In the group of pupils working together on the experiment, two pupils may adjust the weights and the others check the measurements and keep the record. As the experiment proceeds, the pupils may rotate responsibilities so that each has an opportunity to do the complete experiment.

TABLE I

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
w	10	10	10	10	10	10	10	10	10	10
d	12	12	12	12	12	12	12	12	12	12
W	100	20	5	8	15	24	12			
D	12	6	24	15	8	5	10			

As the pupils are doing the experiment and recording measurements in Table I, help them discover two relationships: if the weight is increased, the distance from the fulcrum is decreased; also, if a weight is doubled, the distance from the fulcrum is only half as much; if the weight is made three times as much, the distance is only one-third as much, and so on. The product of weight and distance in each column in Table I should be approximately 120.

Tables II, III, and IV suggest ways of beginning the measurements in order to get a large number of readings and also to help bring out certain relationships. Also, reading the meter stick should increase the pupils' familiarity with this kind of measurement.

TABLE II

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w	16	16	16	16	16	16	16
d	6	6	6	6	6	6	6
W	16	24	6	12	4	8	48
D	6	4	16	8	24	12	2

TABLE III

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w	20	40	10	15	30	$7\frac{1}{2}$	50
d	15	$7\frac{1}{2}$	30	20	10	40	6
W	20	20	20	20	20	20	20
D	15	15	15	15	15	15	15

TABLE IV

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w	18	18	18	18	18	18	18
d	5	5	5	5	5	5	5
W	15	10	30	45	9	6	5
D	6	9	3	2	10	15	18

13-3. Caution: Inductive Reasoning at Work!

After recording all the measurements in their tables, pupils should be able to see and formulate in their own words the law of the lever, namely that the product of one weight and its distance from the fulcrum is equal to the product of the other weight and its distance from the fulcrum. The formula wd = WD should come from the pupils.

Exercises 13-3

1. (a) Equal weights are at equal distances from the fulcrum.

   (b) D is half as much as it was.

   (c) D is twice as much as it was. The new D is twice as large as the original D.

   (d) wd = WD. The product of the measures of the distance and the weight on one side equals the product of the measures of the distance and the weight on the other side.

2. Table V

w	25	35	12	45	21	15	23	11	$14\frac{6}{7}$	14	10	100	100	100
d	3	4	5	3	5	6	4	8	7	5.5	50	50	500	5,000
W	15	20	7	15	14	20	12	$8\frac{4}{5}$	13	10	100	1000	10,000	100,000
D	5	7	$8\frac{4}{7}$	9	$7\frac{1}{2}$	4.5	$7\frac{2}{3}$	10	8	7.7	5	5	5	5

3. Suggest that pupils check the cases with the smallest numbers.

The purpose of Table V is to provide opportunity for pupils to use arithmetic and apply the law to specific cases. The last four figures in the table show that if a sufficiently long lever is used, a tremendous weight can be lifted.

13-4. Graphical Interpretation

Graphing is a skill that needs to be used both in mathematics and science. This experiment gives a set of figures which can be graphed easily.

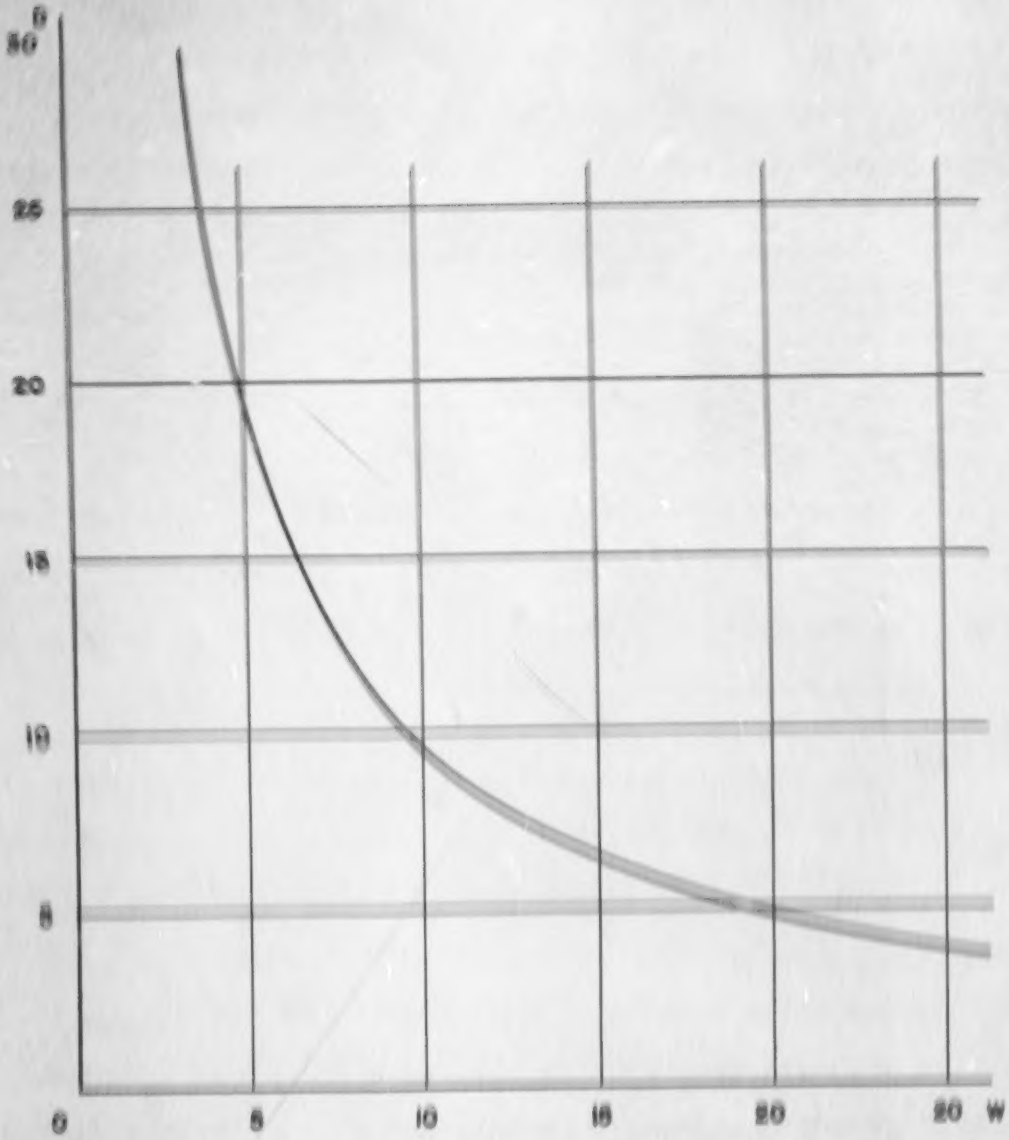
Be sure that the pupils know how to prepare the axes and how to mark the scale on each axis. They may need some preparatory practice in plotting points and probably will find it difficult to draw a smooth curve through the points that they plot.

The points for which one of the pair of numbers is to be found are: (4, 30); (20, 6); (16, $7\frac{1}{2}$); ($6\frac{2}{3}$, 18).

The teacher may wish to follow this graphing with several other graphs, for practice, which yield similar curves, as $WD = 48$ or $WD = 90$. The hyperbola is such an interesting curve that the better students should be encouraged to investigate it and its uses further. It may be appropriate to tell the pupils that the hyperbola is the curve of inverse variation. It results whenever the product of two quantities is a constant as in $WD = 120$, which can also be written $W = \frac{120}{D}$ or $D = \frac{120}{W}$.

Exercises 13-4

1. (a) As the weight increases, the distance decreases.
   (b) As the distance increases, the weight decreases.
   (It may be appropriate to point out that this is an inverse relationship.)
2. (a) ($6\frac{2}{3}$, 18) (b) (8, 15) (c) ($13\frac{1}{3}$, 9)
3. (a) (6, 20) (b) (4, 30) (c) (15, 8)
4. (a) (10, 25) is not. (c) (5, 5) is not.
   (b) (15, 8) is. (d) (20, 15) is not.



GRAPH OF WD = 96
FROM THE DATA OF TABLE II

5. (a) (7, 17.1) (c) (13.3, 9) (e) (5.7, 21)
(b) (8, 15) (d) (17, 7) (f) (23, 5.2)

6. See graph.

7. When W is 20, D is $4\frac{4}{5}$. When D is 12, W is 8.

8. As the weight decreases, the distance increases.
As the weight increases, the distance decreases.

9. The curves have similar shapes. Both show inverse variation.

In all this work, help pupils to become aware of the importance of mathematics in science and of the necessity of mathematical data to carry out experiments.

13-5. Other Kinds of Levers

Other types of levers are simply mentioned in passing. Pupils who are interested may look into them individually but it is not in keeping with the spirit of the chapter to spend class time in a thorough investigation of all types of levers.

13-6. The Role of Mathematics in Scientific Experiment

This discussion of the lever does not introduce new mathematics. The intent here is to suggest in a small way some of the many important roles which mathematics plays in our civilization.

In quantitative studies of all kinds, that is, studies in which quantities are important and are carefully measured, the mathematics of arithmetic and measurement is invaluable. The pupils saw this in measuring distances, counting and comparing weights, and making observations of the data.

In the search for a general pattern or for a new combination of facts, the medium of inductive reasoning is used. Frequently the reasoning process is described in mathematical language which is particularly appropriate for describing general characteristics and properties.

Ordinarily, the general rule itself is described best by a mathematical formula. In interpreting the general pattern which has been described in mathematical language, graphing is often effective. We have seen in our discussions of the lever that the use of a graph gives a convenient picture of a great deal of information.

At many places in the discussion of the lever use was made of deductive reasoning. Sometimes, in fact, it is possible to develop a general law in mathematical terms entirely by using deductive reasoning and mathematical operations based on certain physical assumptions.

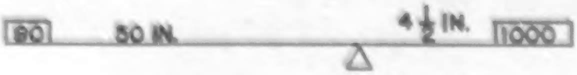
Throughout the entire fibre of our scientific activity and general civilization, mathematics plays a dominant role in analysis, description, interpretation, and generalization.

The historical examples of the text all involve simple mathematics which the pupils should be able to appreciate. You may wish to emphasize the increasingly complex nature of present-day scientific discoveries, many of which use new kinds of advanced mathematics. The class study of the current news of science, as suggested in Chapter I, will probably have revealed a number of good illustrations.

The value of the chapter lies in the understanding which it provides of the way mathematics works with science. There is no particular value in requiring pupils to solve many numerical examples using the principle of the lever unless they need practice in computation. It is difficult to test, in the usual way, the ideas the chapter is designed to present; but a few sample questions for the chapter are included.

Answers to Exercises 13-6

1. Some seesaws are made so that they can be shifted on the fulcrum in order to alter the distances of the riders from the fulcrum and still permit them to sit at the ends of the seesaw. In this way, riders of different weights can balance each other without moving from the ends of the seesaw.
2. 9 yd.
3. 75 pounds
4. $7\frac{1}{2}$ oz.
5. 2 d
6. With a lever on a fulcrum at the proper place, a 90 pound girl can lift 1,000 pounds. For example:



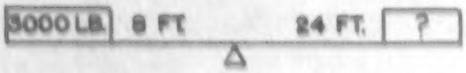
7. About 2 ft. from the fulcrum (1.8 ft.)
8. 15 pounds
9. 1100 pounds
10. (a) and (b)

Sample Test Questions
(Answers at End)

1. If two weights at the opposite ends of a seesaw balance, and one weight is halved, what must be done to the other weight to preserve the balance?
2. If two weights at the opposite sides of a seesaw balance, and one is moved twice as far as it was from the fulcrum, while the other remains unchanged, what happens to the seesaw?

Find the missing value in each case:

3.



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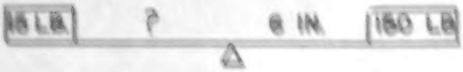

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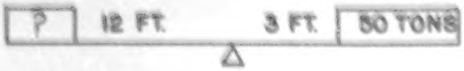

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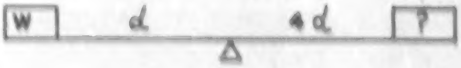

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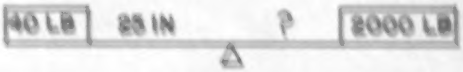

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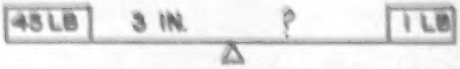

9. Tell how to balance a 60 pound boy and his 40 pound sister if she sits 6 ft. from the center support of the seesaw.

10. Complete the table

w =	9	9	9	9	9	9
d =	8	8	8	8	8	8
W =	2		18		24	
D =		6		16		$\frac{1}{12}$

11. If WD = 30, what happens to D as W increases? What happens to W as D increases?

12. If WD = 75 what is D when W is 15? What is W when D is 10?

13. Which of the points below are on the graph of WD = 63?
(a) (9, 7) (b) (10, 6) (c) (8, 8)

14. State one or two other formulas (similar to WD = 100) in which one value decreases as the other increases.

Answers to Exercises 13-6

(Sample questions)

1. Halve the other weight or move it unchanged so that its distance from the fulcrum is doubled.
2. It is out of balance and the side of the unchanged weight is higher than the other.
3. 1000 lb.
4. 60 inches
5. $12\frac{1}{2}$ tons
6. $\frac{W}{4}$ or $\frac{1}{4}$ W
7. $\frac{1}{2}$ in.
8. 135 in.
9. Seat the boy 4 ft. from the fulcrum.

10

W		12	$4\frac{1}{2}$	864
D	36	4	3	

11. D decreases. W decreases
12. W is 5. D is $7\frac{1}{2}$.
13. (a)
14. rt = 100 lw = 100 vp = 100 (where any constant may be used where 100 appears.)

Time Schedule

Section	
1-2	1
2-3	1
4	1
5-6	1

Bibliography

1. Newman, James R. THE WORLD OF MATHEMATICS. New York: Simon and Schuster, 1956.

   (Galileo, pp. 726-770; falling bodies, p. 729; pendulum, pp. 729, 741; Eratosthenes, pp. 205-207.)

2. Kline, Morris. MATHEMATICS IN WESTERN CULTURE. New York: Oxford University Press, 1953.

   Chapter XIII The Quantitative Approach to Nature
   Chapter XIV The Deduction of Universal Laws